On the Accuracy of Semi-Lagrangian Numerical Simulation of Internal Gravity Wave Motion in the Atmosphere

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Abstract

We have investigated the accuracy of the semi-implicit semi-Lagrangian (SISL) method in simulating internal gravity wave (IGW) motion. We have focused on the relative accuracy of the hydrostatic, and nonhydrostatic IGW solutions. The analysis is based on a linearized model and a Global Circulation Model-Dynamic Core (GCM-DC) with a stretched grid.

The nonhydrostatic version of the GCM-DC model produces the familiar IGW train disturbance anchored to an isolated hypothetical mountain. The wave has a distinct tilt away from the vertical direction, which is consistent with classical theory. For the hydrostatic version of the model, the axis of the resulting IGW train rests nearly perpendicular to the mountain top, thus again consistent with classical theory. Increasing the time step from 10 s; Courant number \((Cn) = 0.5\); to 60 s \((Cn = 3.0)\), results in stable solutions for both the hydrostatic and nonhydrostatic versions of the model. The nonhydrostatic solution is in close agreement with the control run however, the hydrostatic solution exhibits large phase truncation errors.

The solutions for the one-dimensional linearized SISL model confirm the GCM-DC results that the nonhydrostatic IGW train is less damped and shifted by the SISL scheme than the corresponding hydrostatic IGW motion. The linear solutions indicate very high accuracy of the physical mode of the solution, but it rapidly deteriorates when \(Cn\) exceeds unity. As \(\Delta t \rightarrow 0\) the amplitude of the computational mode tends to zero and its frequency to infinity. However, as \(\Delta t \rightarrow \infty\), the frequency of the computational SISL mode asymptotically approaches the value of the frequency of the corresponding SISL physical mode. Furthermore, the amplitude of the SISL computational mode is directly proportional to the size of the time step. Therefore, at large time steps, the amplification of the computational mode could offset some of the numerical damping of the physical mode by the SISL scheme.

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1. Introduction

Comprehensive understanding of the behavior of truncation errors of numerical weather/climate models is important for accurate interpretation of the resulting simulations. Explicit Eulerian numerical integration schemes suffer from the need to use prohibitively small time steps. The maximum timestep is limited by the speed of the fastest propagating atmospheric waves. Consequently, the truncation errors in time are much smaller than the truncation errors in space, and the latter dominates the total error (Semazzi and Dekker 1994). The semi-implicit Eulerian schemes are more efficient than the explicit Eulerian method because the size of the time step is limited by the magnitude of the advection velocity, which is typically an order of magnitude less than the propagation speed of the fasted waves.

Ever since Robert (1981, 1982) introduced the semi-implicit semi-Lagrangian (SISL) numerical method for the integration of atmospheric prediction models, it has steadily gained in popularity (Bates et al. 1990, 1993; Côté et al. 1998a,b; Côté and Staniforth 1998; McDonald and Haugen 1992; Moorthi et al. 1994; Pouliot 2000; Staniforth et al. 1994; Tanguay et al. 1990; Yeh et al. 2002; Steppler and Herbert 2003). The SISL scheme is unconditionally stable and the size of the time step is limited by accuracy rather than stability. Therefore, even within the same formal order of accuracy, model performance may vary considerably (Semazzi et al. 1996). For synoptic and planetary scale wave motion there is a finite optimal SISL model time step which yields maximum accuracy (Bates 1984; Semazzi and Dekker 1994; Semazzi et al. 1996).

In the case of high-frequency gravity wave motion generated by orography, several studies have found different kind of relationships between the size of time step and the quality of the numerical solutions. Simmons et al. (1978) found that semi-implicit methods could encounter numerical instabilities when used in the integration of high-frequency gravity waves generated by orography. Using a SISL numerical model Pinty et al. (1995) analysed flow over a hypothetical mountain by comparing the numerical results with classical solutions for various mountain-wave regimes. They obtained accurate representation of forced hydrostatic gravity waves for Courant numbers less than 0.5. Equally significant, they noted that in the case of nonhydrostatic flows this limitation may be less severe. Hereil and Laprise (1996) found that although the SISL method is unconditionally stable, when the Courant number is greater than one wrong stationary internal gravity wave, solutions may be obtained in the presence of bottom topographic forcing. Using a nonhydrostatic meso-scale model with grid mesh size of about 3 km, Bartello and Thomas (1996) noted that Courant numbers considerably less than one may be required to ensure acceptable accuracy. In a study based on the NCEP meso-scale Eulerian “Eta” model, Janjic et al. (2001) have pointed out that their nonhydrostatic version of the model generates less spurious noise, and therefore requires less dissipation, than the corresponding hydrostatic version to control computational instabilities. Therefore, based on previous studies there is good evidence indicating that the hydrostatic formulation is more susceptible to elevated truncation error occurrences than the nonhydrostatic models for meso-scale motion.

The objective in this study is to investigate the accuracy of the GCM-DC model in simulating meso-scale internal gravity wave motion. We focus on the relative accuracy in the simulation of hydrostatic and nonhydrostatic internal gravity wave motion. The class of nonhydrostatic solutions investigated recently by Qian and Kasahara (2003) is on a much larger scale and in the future, it would be important to undertake a separate study to examine the performance of the SISL method in simulating that kind of motion.

The description of the global semi-implicit semi-Lagrangian model (Global Circulation Model-Dynamic Core ‘GCM-DC’) used in the investigation is given in section 2. The numerical results based on the GCM-DC are described in section 3. The linearized SISL model, and the construction of its solutions, are described in section 4. Section 5 deals with the discussion of the linear model solutions. Conclusions are presented in section 6.

2. Description of the GCM-DC model

Details of the SISL formulation of the model are described in Semazzi et al. (1995), Qian
(1998a), and Pouliot et al. (2000). The customized elliptic equation solver used to calculate the mass field of the model is described in Settumba (2000). In this section we only briefly summarize the salient features of the model and the recent modifications in its formulation.

**Attributes of the GCM-DC model:** The primary attributes of the model may be summarized as follows: (i) fully compressible non-hydrostatic system of equations with bottom topography, (ii) direct discretization of the 3-dimensional vector momentum equation, (iii) three-dimensional semi-Lagrangian trajectories, (iv) second-order trajectory uncentering, to suppress computational noise, (v) tri-linear interpolation for locating the departure points, and tri-cubic interpolation for obtaining the values at the departure points, (vi) two-time level semi-implicit semi-Lagrangian advection scheme with first-order accuracy in time and level semi-implicit semi-Lagrangian advection values at the departure points, (vii) isothermal scheme with first-order accuracy in space, (viii) fully compressible non-hydrostatic system of equations with bottom orography, (ix) direct discretization of the 3-dimensional vector momentum equation, (x) a stretched Arakawa C-grid in spherical coordinates with coordinate rotation near the poles to reduce numerical truncation errors associated with the Earth’s curvature in the computation of the trajectories and departure points for the air parcels, (xi) Gal-Chen and Somerville height-based terrain following vertical coordinate system with 14 coordinate levels, (xii) Charney-Phillips vertical staggered grid, (xiii) the Jacobi elliptic equation solver for the mass variable, which was used in Qian et al. (1998a) has been replaced with a more robust customized solver, based on the Generalized Minimum Residual Method (GMRES) with a block Jacobi preconditioner (Settumba 2000), and (xiv) a sponge layer inserted below the top rigid lid of the model to absorb incident wave energy following Qian et al. (1998b).

**Stretched coordinate system:** The uniform grid employed in Semazzi et al. (1995), and Qian et al. (1998a) has been replaced by a stretched grid (Fox-Robinovitz et al. 1997; Qian et al. 1998b). It is defined in terms of a local stretching ratio,

\[ r_j = \Delta x_j/\Delta x_{j-1}. \]  

(2.1)

for the adjacent grid intervals; \( x_j \) is the distance over which the grid is to be stretched, \( j \) is the number of grid points involved in the stretching. For a uniform grid, \( r_j = 1 \). In the case of a uniformly stretched grid \( r_j = \text{constant} > 1 \) for all \( j \)'s. Another important ratio used to construct the grid is, \( R = \Delta x_{\text{max}}/\Delta x_{\text{min}} \), that expresses the relative size of the largest to the smallest grid intervals for the entire spatial numerical domain. In other words, the resolution is uniform \( (\Delta x_0) \) over the high resolution region, and is progressively degraded with increasing distance from the edge of the ‘zoom’ region. A coordinate transformation can be derived for a uniformly stretched grid \( (r_j = \text{constant}) \) from,

\[ \Delta x_j = r^j \Delta x_0. \]  

(2.2)

The coordinates for the stretched grid points are given by,

\[ x_j = \Delta x_0(1 + r + \cdots + r^{j-1}) = \Delta x_0 \frac{r^j - 1}{r - 1} \]  

(2.3)

where \( \Delta x_0 \) is the first grid interval starting from which the grid is stretching rightward in the original coordinate system. The local stretching ratios are calculated as a root, \( r \), of the following equation,

\[ r^j - \frac{x_j}{\Delta x_0} r + \frac{x_j}{\Delta x_0} - 1 = 0, \]  

(2.4)

After the local stretching ratio is determined the grid is generated using (2.3).

**Bottom orography:** The idealized topography centered at the equator is a three-dimensional bell-shaped Witch of Agnesi mountain,

\[ Z_s(\lambda, \theta) = \frac{h_0 d^2}{d^2 + [\frac{a(\lambda - \lambda_0) \cos \theta]^2 + [\frac{a(\theta - \theta_0)]^2}, \]  

(2.5)

where, \( h_0 \) is the topographic height, \( d \) is the half-width of the mountain, \( (\lambda_0, \theta_0) \) is the longitude and latitude of the mountain top and \( a \) is the radius of the Earth. We generated a variable resolution grid with grid size of 2 km over the region surrounding the isolated mountain (2.5). This yields a grid with a large global stretching ratio \( (R) \) that could result in undesirable numerical effects, such as wave reflec-
tion (Fox-Robinowitz et al. 1997). However, this presents no practical problems in our specific situation since we are primarily interested in comparing the hydrostatic and nonhydrostatic response to orographic forcing near the mountain.

3. GCM-DC simulation results

a. Control simulation ($\Delta x = \Delta y = 400 \text{ m}, \Delta t = 10 \text{ s, and } U = 20 \text{ ms}^{-1}$)

A pair of nonhydrostatic and hydrostatic simulations, with a 10 s timestep corresponding to a Courant number ($C = U_0 \Delta t / \Delta x$) of 0.5 have been performed. In Fig. 1 we show the numerical results for vertical motion $w$ generated by both the hydrostatic and nonhydrostatic versions of the model. The former produces an internal gravity wave train extending vertically upward from the mountain top, with only slight tilt from the vertical direction. However, the corresponding solution for the nonhydrostatic version of the model has a distinct tilt downstream of the isolated mountain. This difference in the structure of the two solutions, in the quasi-linear regime, is well known and consistent with the classical analytical solutions (Smith 1980).

b. Large Courant number ($\Delta x = \Delta y = 400 \text{ m}, \Delta t = 60 \text{ s, and } U = 20 \text{ ms}^{-1}$)

For the second pair of simulations based on the GCM-DC model, the Courant number was increased from 0.5 to 3.0 by increasing the time step from 10 s to 60 s. As in the control experiment (Fig. 1), the nonhydrostatic/hydrostatic binary switch was used to obtain the corresponding two sets of solutions. The results for the variation of the vertical motion in the longitude-height cross-section are displayed in Fig. 2. The simulations are stable in spite of the very large Courant number that was used.

The main result in this pair of experiments is that the hydrostatic version of the model produces a tilted gravity wave train, thus inconsistent with the classical analytical solutions and the numerical solution obtained with a modest Courant number of 0.5 in Fig. 1. It is apparent that high computational efficiency may be achieved at the cost of significant loss of accuracy. This outcome holds for the full range of the second-order trajectory uncentering parameter, $e_s$ (defined in 4.7–4.9).

4. Linear Semi-implicit semi-Lagrangian model

To further investigate the peculiar behavior exhibited by the semi-Lagrangian GCM-DC solutions for large time steps in section 3, in this section we examine the numerical solutions of a linearized semi-Lagrangian model. The corresponding analytic solutions are presented in appendix A.

We have adopted the linearized form of

\[
w (\text{m/s}) \text{ hydrostatic (400 m resolution)}
\]

\[
w (\text{m/s}) \text{ nonhydrostatic (400 m resolution)}
\]

Fig. 1. Control idealized orography experiment ($\Delta x = \Delta y = 400 \text{ m, } \Delta t = 10 \text{ s, and } U = 20 \text{ ms}^{-1}$). The second order uncentering parameter $e_s = 0.5$, and Courant number $= 0.5$. Cross-section of GCM-DC model vertical velocity at the Equator for, (a) hydrostatic, and (b) nonhydrostatic model solutions after 6 hrs. An idealized bell-shaped Witch of Agnesi mountain, with half-width of 2 km and maximum height of 500 m has been placed at the bottom of the atmosphere. Contour interval is 0.5 m/s.
Euler’s equations which permits both hydrostatic and nonhydrostatic solutions. This formulation is similar to the form used by Qian and Kasahara (2003) except that in our case the basic state background flow \( \mathbf{u}_0 \), and there is no dependency of the model variables on the meridional direction:

**Horizontal motion equation:**

\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \frac{\partial \phi}{\partial x},
\]

(4.1)

**Vertical motion equation:**

\[
\left( \frac{\partial w}{\partial t} + \mathbf{u} \cdot \nabla w \right) \delta = -\frac{1}{\rho} \frac{\partial \phi}{\partial z},
\]

(4.2)

**Thermodynamics equation:**

\[
\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = \rho \Lambda^2 w \frac{1}{g},
\]

(4.3)

**Mass conservation equation:**

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho \mathbf{u}}{\partial x} = -\Gamma \frac{\partial \rho}{\partial z} w,
\]

(4.4)

To obtain the finite difference form of this system of equations we shall replace the partial derivative in space (x-coordinate), and the local time tendency using the following identity, \( d\Psi/dt \equiv \partial \Psi/\partial t + \mathbf{u} \cdot \nabla \Psi/\partial x \).

The symbols used in the linear analysis are defined in appendix C. To facilitate sensitivity calculations for addressing the key question posed in this investigation, we have introduced two binary switches. In the vertical momentum equation (4.2), \( \delta \) is equal to one for the non-hydrostatic dynamics and zero for the hydrostatic conditions. In the mass conservation equation (4.4), \( \Gamma \) is equal to zero for a Boussinesq stratified atmosphere, otherwise it is equal to one. The basic state satisfies hydrostatic balance thus,

\[
\frac{d\bar{\rho}}{dz} = -\bar{\rho} g,
\]

(4.5)

and \( \Lambda \) is the Brunt-Vaisala frequency of the basic state. For an isothermal atmosphere with \( \Lambda \approx 0.02 \text{ s}^{-1} \) the buoyancy period is about 5 minutes (Nappo 2002). In our computations we have assumed that \( \Lambda = 0.012 \text{ s}^{-1} \) (buoyancy period \( \approx 8 \text{ minutes} \)), which is typical for average tropospheric conditions (Holton 1992).

**Semi-implicit semi-Lagrangian scheme:** To apply the SISL finite difference integration scheme the governing equations are expressed as follows,

\[
\frac{d\psi}{dt} \approx \frac{D\psi}{Dt} = H_l + H_{nl},
\]

(4.6)

where \( (\ )_l \) and \( (\ )_{nl} \) refer to the linear and nonlinear terms, respectively. As in Semazzi et al. (1995) and other references therein, the forecast is performed by integrating (4.6) along the air parcel backward trajectory using a two-time level scheme (Temperton and Staniforth 1987),
in which case,
\[ \psi - \psi_0 = \Delta t (\dot{H}_l + \dot{H}_{nl}), \]  
where,
\[ \dot{H}_l = 2H_l(x, y, t + \Delta t) + \beta H_l(x_s, y_s, t) \]
\[ + \gamma H_l(x_{ss}, y_{ss}, t - \Delta t), \]
\[ \dot{H}_{nl} = 2H_{nl}(x, y, t) + \beta H_{nl}(x_s, y_s, t) \]
\[ + \gamma H_{nl}(x_{ss}, y_{ss}, t - \Delta t), \]
\[ \dot{H}_{nl} = 2H_{nl}(x, y, t) - H_{nl}(x, y, t - \Delta t), \]
with,
\[ \alpha = \frac{1 + \varepsilon_s}{2}, \quad \beta = \frac{1 - 2\varepsilon_s}{2}, \quad \gamma = \frac{\varepsilon_s}{2}, \]  
where \((\varepsilon_s)\) is the second order uncentering coefficient. The uncentering is introduced to understand its effects on the accuracy of the SISL in simulating internal gravity waves. In more comprehensive nonlinear models, the purpose of uncentering is to suppress the undesired amplification of spurious noise associated with orographic forcing (Tanguay et al. 1992; Rivest et al. 1994). The single asterisk denotes the locations of the air particles at time \((t)\), while the double asterisk denotes upstream locations at time \((t - \Delta t)\). The departure point values are located using the following formula,
\[ x_s = x - \pi \Delta t, \quad x_{ss} = x_s - \pi \Delta t. \]  
The corresponding expressions in the \((y)\) and \((z)\) direction are not applicable here because there is no motion at all in the \((y)\) direction, and no Lagrangian advection in the vertical because \(\dot{W} = 0\).

To obtain the SISL finite difference approximations to (4.1) through (4.4), we apply (4.7) to each equation. The second term on the right-hand-side of (4.7) is neglected because all the terms in the model are linear. For the GCM-DC model discussed in sections 2 and 3 the nonlinear terms are nonzero. The finite difference equations for the linearized system may be stated as follows,
\[ u_{1.K.N+1} - u_{s,N} \]
\[ = -\Delta t \left\{ \alpha \left( \frac{1}{\rho} \frac{\partial \phi}{\partial x} \right)_{1.K.N+1} + \beta \left( \frac{1}{\rho} \frac{\partial \phi}{\partial x} \right)_{s,N} \right\} \]
\[ + \gamma \left( \frac{1}{\rho} \frac{\partial \phi}{\partial x} \right)_{s,N-1} \],  
\[ \left( w_{1.K.N+1} - w_{s,N} \right) \delta \]
\[ = -\Delta t \left\{ \alpha \left( \frac{1}{\rho} \frac{\partial \phi}{\partial z} \right)_{1.K.N+1} + \frac{\rho}{\rho} \right\} \]
\[ + \beta \left( \frac{1}{\rho} \frac{\partial \phi}{\partial z} \right)_{s,N} \}
\[ + \gamma \left( \frac{1}{\rho} \frac{\partial \phi}{\partial z} \right)_{s,N-1} \}
\[ \rho_{1.K.N+1} - \rho_{s,N} \]
\[ = \Delta t \left\{ \alpha \left( \frac{\rho \Lambda^2}{g} \right) w_{1.K.N+1} \right\} + \frac{\rho \Lambda^2}{g} \left( w_{s,N} \right) \}
\[ + \gamma \left( \frac{\rho \Lambda^2}{g} \right) w_{s,N-1} \}
\[ \left( \frac{\partial u}{\partial x} \right)_{1.K.N} + \frac{\partial u}{\partial z} \left( \frac{\partial \phi}{\partial x} \right)_{1.K.N} = -\frac{\partial \rho}{\partial z} u_{1.K.N} \]

Polynomial interpolation of a model variable(s) at the departure points is based on (4.15) and (4.16). The interpolation weights are based on Carnahan et al. (1969; also see Table 1). Comprehensive analysis of the performance of various interpolation schemes is given by McDonald (1999).
\[ s_{1.K.N} = \sum_r C_r s_{1-P-r,K.N} \]  
\[ s_{s,K.N} = \sum_r C_r s_{s-P-s-r,K,N-1} \]

The range of \(\hat{z}\) for the linear and cubic interpolation schemes is \(0 \leq \hat{z} \leq 1\), and \(-0.5 \leq \hat{z} \leq 0.5\) for the quadratic case. An air parcel that arrives at a grid point anywhere on grid surface (K) originates from the same level since \(\dot{W} = 0\). Therefore, no vertical interpolation is required to compute the values at the departure locations. Substituting (4.15) and (4.16), into (4.11) through (4.14), and applying second order accurate centered spatial differencing yields,
Derivation of SISL numerical solutions for the linear model: We assume the usual harmonic dependency (Nappo 2002) and write,

\[
\begin{align*}
(u, w) & = \lambda_{\text{SISL}}^N \begin{pmatrix} \tilde{u} \cdot e^{i(k_\Delta x + K(m-iq)\Delta z)} \\ \rho, \phi \end{pmatrix}_{I,K,N} \\
\therefore (u, w) & = \lambda_{\text{SISL}}^N \begin{pmatrix} \tilde{u} \cdot e^{i(k_\Delta x + K(m-iq)\Delta z)} \\ \rho, \phi \end{pmatrix}_{I,K,N},
\end{align*}
\]

(4.21)

We define, \( q = \frac{g}{RT} \), where, \( T_0 \) is the temperature of the isothermal basic state. Other symbols are defined in appendix C. Substituting (4.21) into (4.17) through (4.20), introducing the following notation,

\[
\Psi = \lambda_{\text{SISL}}^2 - \lambda_{\text{SISL}} \sum_r C_r e^{-ik(P+r)\Delta x},
\]

(4.22)

\[
R = \lambda_{\text{SISL}}^2 + \beta \lambda_{\text{SISL}} \sum_r C_r e^{-ik(P+r)\Delta x}
\]

(4.23)

and dividing all the terms of the resulting system of equations by \( \lambda_{\text{SISL}}^{-1} e^{i(k_\Delta x + K(m-iq)\Delta z)} \), we obtain,

\[
\Psi \tilde{u} = -\frac{\Delta t}{\rho} \left( \frac{e^{i(k_\Delta x - e^{-ik_\Delta x}}}{2\Delta x} \right) e^{-2Kq_\Delta z} \tilde{u},
\]

(4.24)

\[
\Psi \tilde{w} = -\left( \frac{\Delta t}{\rho} \left( \frac{e^{i(m+iq)\Delta z} e^{-i(m+iq)\Delta z}}{2\Delta z} \right) \right) e^{-2Kq_\Delta z} \tilde{w} + g\rho \tilde{w} e^{-2Kq_\Delta z} R,
\]

(4.25)

\[
\Psi e^{-2Kq_\Delta z} \tilde{w} = \left( \frac{\rho \Delta t}{g} \right) \tilde{w} R,
\]

(4.26)

\[
u \left( \frac{e^{i(k_\Delta x - e^{-i(k_\Delta x)}}}{2\Delta x} \right) + \tilde{w} \left( \frac{e^{i(m+iq)\Delta z} e^{-i(m+iq)\Delta z}}{2\Delta z} \right)
\]

(4.27)

---

**Table 1. Interpolation weights for the linear, quadratic, and cubic schemes**

<table>
<thead>
<tr>
<th>Summation index</th>
<th>Linear Scheme</th>
<th>Quadratic Scheme</th>
<th>Cubic Scheme</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r = -1 )</td>
<td>( C_{r-1}^* = 0 )</td>
<td>( C_{r-1}^* = -\frac{1}{2} \hat{z}^* (1 - \hat{z}^*) )</td>
<td>( C_{r-1}^* = -\frac{1}{2} \hat{z}^* (1 - \hat{z}^<em>) (2 - \hat{z}^</em>) )</td>
</tr>
<tr>
<td>( r = 0 )</td>
<td>( C_0^* = (1 - \hat{z}^*) )</td>
<td>( C_0^* = (1 - \hat{z}^<em>) (1 + \hat{z}^</em>) )</td>
<td>( C_0^* = \frac{1}{2} (1 - \hat{z}^<em>) (1 + \hat{z}^</em>) (2 - \hat{z}^*) )</td>
</tr>
<tr>
<td>( r = 1 )</td>
<td>( C_1^* = \hat{z}^* )</td>
<td>( C_1^* = \frac{1}{2} \hat{z}^* (1 + \hat{z}^*) )</td>
<td>( C_1^* = \frac{1}{2} \hat{z}^* (1 + \hat{z}^<em>) (2 - \hat{z}^</em>) )</td>
</tr>
<tr>
<td>( r = 2 )</td>
<td>( C_2^* = 0 )</td>
<td>( C_2^* = 0 )</td>
<td>( C_2^* = -\frac{1}{2} \hat{z}^* (1 - \hat{z}^<em>) (1 + \hat{z}^</em>) )</td>
</tr>
</tbody>
</table>

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We apply the identities
\[
(e^{i\Delta x} - e^{-i\Delta x})/2 = i \sin k\Delta x;
\]
\[
(e^{i(m \pm iq)\Delta x} - e^{-i(m \pm iq)\Delta x})/2 = i \sin(m \pm iq)\Delta z
\]
to (4.24) through (4.27), and express the result in matrix form as follows,
\[
\begin{pmatrix}
\psi \\
0 \\
\psi \\
0 \\
\frac{(\sin k\Delta x)}{\Delta x} \\
\frac{(\sin(m - iq)\Delta x)}{\Delta x} \\
\frac{(\sin(m + iq)\Delta x)}{\Delta x}
\end{pmatrix}
\begin{pmatrix}
\frac{\phi L}{\Delta x} \\
\phi L \\
\phi L \\
\frac{(\sin k\Delta x)}{\Delta x} \\
\frac{(\sin(m - iq)\Delta x)}{\Delta x} \\
\frac{(\sin(m + iq)\Delta x)}{\Delta x}
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
= (4.28)
\]

For non-trivial solutions of the eigenvalue problem, the matrix on the left hand side of (4.28) must be singular, thus its determinant must be identically equal to zero. Application of this condition yields,
\[
\frac{\sin k\Delta x}{\Delta x} = \frac{\sin(m - iq)\Delta x}{\Delta x} = \frac{\sin(m + iq)\Delta x}{\Delta x},
\]
\[
\frac{(\sin(m + iq)\Delta x)}{\Delta x} = \frac{(\sin k\Delta x)}{\Delta x},
\]
\[
\frac{(\sin(m - iq)\Delta x)}{\Delta x} = \frac{(\sin k\Delta x)}{\Delta x}.
\]
Therefore (4.29) becomes,
\[
(\delta - \tau - i\eta)\Psi^2 + (\Delta t)^2 \Lambda^2 R^2 = 0.
\]

The terms $\Psi^2$ and $R^2$ may be obtained by squaring (4.22) and (4.23). The result is,
\[
\Psi^2 = \lambda^4_{SISL} - 2\Theta \lambda^3_{SISL} + \Theta^2 \lambda^2_{SISL},
\]
and,
\[
R^2 = \lambda^2_{SISL} + 2\lambda_{SISL} \Theta^2 + \Theta^2 \lambda^2_{SISL} + 2\Theta \lambda_{SISL} + \gamma^2 \Omega^2,
\]
where
\[
\Theta = \sum \beta e^{-i(k(P' - r)\Delta x)}
\]
and
\[
\Omega = \sum \gamma e^{-i(k(P' - r)\Delta x)}.
\]
Therefore (4.33) results in a fourth-degree polynomial equation in $\lambda$ of the form,
\[
\mu_4 \lambda^4_{SISL} + \mu_3 \lambda^3_{SISL} + \mu_2 \lambda^2_{SISL} + \mu_1 \lambda_{SISL} + \mu_0 = 0
\]
where the complex coefficients are defined as follows,
\[
\mu_4 = \{\delta + (\Delta t)^2 \Lambda^2 \lambda^2 - (\tau + i\eta)\}
\]
\[
\mu_3 = \{2(\Delta t)^2 \Lambda^2 2\beta \Theta - 2\Theta \lambda (\tau + i\eta)\}
\]
\[
\mu_2 = \{\delta \Theta^2 + (\Delta t)^2 \Lambda^2 (\beta^2 \Theta^2 + 2\lambda \Omega) - \Theta^2 (\tau + i\eta)\}
\]
\[
\mu_1 = \{(\Delta t)^2 \Lambda^2 \lambda \gamma \Theta \Omega\}
\]
\[
\mu_0 = (\Delta t)^2 \Lambda^2 \gamma^2 \Omega^2
\]

5. Discussion of the linear model solutions

5.1 Linear model quartic solutions (second order trajectory uncentering)

First, we discuss the more general linear model solutions based on the second-order trajectory uncentering scheme. In this case there are four roots for (4.37), of which two are physical modes (upward and downward propagating internal gravity wave trains), and the other two correspond to a couplet of computational modes also moving in opposite directions from each other. For all the solutions discussed in this
paper we assume Boussinesq atmospheric conditions by setting $\Gamma = 0$. Spiegel and Veronis (1960) have shown that in a fluid based on the Boussinesq approximation, the perturbations in density associated with local pressure changes are negligible and acoustic waves are eliminated.

The solutions to (4.37) may be obtained by using any polynomial equations solver software, readily available in standard computational libraries. In our case, we employed the Maple software to obtain the desired solutions. Although this approach is efficient it provides no flexibility for close analysis of the functional mathematical form of the linear solutions since it is applied as a black-box. In section 5.2 we shall examine the mathematical form of the solutions.

**Hydrostatic regime ($L_x = 20 \text{ km}$):** In this set of calculations we examine the solutions when the horizontal wavelength $L_x$ is 20 km, thus corresponding to relatively large-scale motion over the mesoscale wave spectrum.

Figure 3 displays frequency as a function of the size of the time step. The downward propagating wave solutions and computational modes are not shown in the figure. The solutions for the upward propagating IGWs correspond to, (i) the physical hydrostatic SISL solution, (ii) the physical nonhydrostatic SISL solution, (iii) the hydrostatic analytic solution and, (iv) the nonhydrostatic analytic solution. The small difference that exists between the analytical and numerical solutions, for $L_x = 20 \text{ km}$, is due to spatial truncation errors. Inspection of (4.32) indicates that the spatial truncation error in the horizontal and vertical directions tend to cancel each other when the scale and resolution are the same in both directions. This explains why the spatial truncation error for $L_x = 5 \text{ km}$ is virtually zero and the fact that it is significantly less than the corresponding error for $L_x = 20 \text{ km}$.

**Nonhydrostatic regime ($L_x = 5 \text{ km}$):** Figure 3 shows that at the much smaller scale of 5 km the frequencies for hydrostatic and non-
hydrostatic IGW motion are significantly different, thus consistent with the analytic solutions. It is instructive to determine why the hydrostatic internal gravity frequency ($\omega_{0.012 \text{ s}^{-1}}$) is virtually the same as the Brunt-Vaisala frequency ($\Lambda$). We examine the analytical solution (A3) to see how this arises. For the present calculations we chose $\delta = m$ (i.e., $L_x = L_z = 5 \text{ km}$). First, we focus on the hydrostatic case thus, $(\delta = 0)$. For simplicity we also assume motionless background motion and let, $\delta k = 0$. Following (Nappo 2002), we may neglect the term involving $q$, since $q \ll k$. With these considerations (A3) reduces to $\omega_{0.012} \approx \Lambda$, thus in agreement with $\omega_{SISL}$ as $\Delta t \rightarrow 0$ in Fig. 3. Adopting similar reasoning for the nonhydrostatic case (i.e., $\delta = 1$) we find that $\omega_{\Lambda/\sqrt{2}} = 0.0085 \text{ s}^{-1}$. Again, it is reassuring to note that this is in agreement with the value of $\omega_{SISL}$ as $\Delta t \rightarrow 0$ in Fig. 3. At sufficiently large time steps greater than 60 s (not shown), the graphs for the hydrostatic and nonhydrostatic solutions eventually intersect, despite the fact that $\omega_{\Lambda(\text{nonhydrostatic})} < \omega_{\Lambda(\text{hydrostatic})}$. It is therefore evident that besides representing motion at short scales more realistically, the SISL nonhydrostatic model also results in more accurate solutions. We also note that at Courant number of approximately one, the truncation error due to temporal discretization, exceeds the corresponding error due to spatial discretization. It is therefore apparent, that in order to maximize the benefits from the unconditional stability of the SISL scheme, it would be desirable to adopt a SISL scheme with higher order accuracy than the first-order scheme employed in this and many other SISL numerical schemes. The forward trajectory transport schemes, such as Nair et al. (2003), is promising in this respect, because it also conserves mass.

The complete set of numerical solutions for frequency for both the hydrostatic and nonhydrostatic versions of the model is shown in Fig. 4. Altogether, there are eight solutions,
four corresponding to the hydrostatic model, and four to the nonhydrostatic version. Negative frequencies are associated with downward propagating wave motion. The positive sign, which we are more interested in, gives the frequencies for the 4 upward propagating waves. As we can see the frequencies for the computational modes are significantly larger than those of the physical modes at small timesteps less than 60 s. For larger time steps greater than 60 s (not shown), the magnitude of the frequencies for the computational modes drop dramatically, and become comparable to those for the physical modes.

The magnitude of the corresponding amplitudes for the physical and computational solutions are computed from (B2) and (B1). They are shown in Fig. 5 (|λ_{SISL(3,4)}|), and Fig. 6 (|λ_{SISL(1,2)}|), respectively. As in the case of the corresponding frequency solutions in Fig. 3, the amplitudes for the physical mode for the nonhydrostatic model are more accurate than in the case of the hydrostatic model. The amplitude of the computational modes grows linearly with the size of the time step, and that of the computational nonhydrostatic wave solution is smaller than for the corresponding magnitude for the hydrostatic model solution.

As Δt → ∞, extension of Fig. 4 (not shown) indicates that the frequency of the computational SISL mode asymptotically approaches the value of the frequency for the corresponding physical SISL mode. Since the amplitude for the computational SISL mode is directly proportional to the size of time step (Fig. 6), the results suggest that the resulting amplification may have positive contribution of offsetting some of the damping (Fig. 5) associated with the SISL physical mode, particularly at large time steps.

5.2 Linear model quadratic solutions (first order trajectory uncentering)

a. Derivation of mathematical formula for the numerical solutions

In this section we discuss the solutions of the linear model, based on the first order trajectory uncentering scheme. There are only two solu-

![Graph](image-url)

Fig. 5. Physical hydrostatic (|λ_{SISL(3,4)}|, δ = 0) and nonhydrostatic (|λ_{SISL(3,4)}|, δ = 1) solutions for the linear model; vertical scale $L_z = 5$ km; horizontal scale $L_x = 5$ km; vertical resolution $Δz = 500$ m, horizontal resolution $Δx = 500$ m; Brunt-Vaisala frequency = 0.012; second order uncentering parameter $e_x = 0.5$; $Δ_{CFL}^{-1} = 20$ s. Continuous/dashed line styles correspond to nonhydrostatic (NH)/hydrostatic (H), solutions.
tions in this case, corresponding to the upward and downward propagating internal gravity waves. Again we assume that Boussinesq conditions apply, and set $G = 0$ in (4.37). For the first-order trajectory uncentering scheme the parameters of the second-order trajectory uncentering scheme in (4.9) reduces to the form,

$$a = \left(1 + e_f \right)/2, \quad b = \left(1 - e_f \right)/2, \quad \text{and} \quad g = 0.$$  \hspace{1cm} (4.39)

Therefore the solution comprises two physical modes and no computational modes since only two time levels ($t$ and $t + \Delta t$) are involved. With these assumptions, $\mu_0 = \mu_1 = 0$ in (4.37) and the resulting polynomial in $\lambda$ is a quadratic of form,

$$\mu_4 \lambda_{SISL}^2 + \mu_3 \lambda_{SISL} + \mu_2 = 0,$$  \hspace{1cm} (4.40)

where the constant coefficients are defined by,

$$\mu_4 = \{\delta + (\Delta t)^2 \Lambda^2 \beta^2 - \tau\},$$  \hspace{1cm} (4.41)

$$\mu_3 = 2\Theta\{(\Delta t)^2 \Lambda^2 \beta - \delta + \tau\},$$  \hspace{1cm} (4.41)

$$\mu_2 = \Theta^2\{\delta + (\Delta t)^2 \Lambda^2 \beta^2 - \tau\}.$$  \hspace{1cm} (4.41)

The roots of (4.40) are obtained by applying the usual quadratic formula. After considerable algebra we find that,

$$\lambda_{SISL} = \Theta\left\{\frac{[\delta - (\Delta t)^2 \Lambda^2 \beta - \tau] \pm \Delta t \Lambda \sqrt{(\tau - \delta)(\tau + \beta)^2}}{[\delta + (\Delta t)^2 \Lambda^2 \beta^2 - \tau]}\right\},$$  \hspace{1cm} (4.42)

and since $(\alpha + \beta) = \frac{1+\nu}{2} + \frac{1-\nu}{2} \equiv 1$, (4.42) becomes,

$$\lambda_{SISL} = \Theta\left\{\frac{[\delta - (\Delta t)^2 \Lambda^2 \beta - \tau] \pm i\Delta t \Lambda \sqrt{\delta - \tau}}{[\delta + (\Delta t)^2 \Lambda^2 \beta^2 - \tau]}\right\},$$  \hspace{1cm} (4.43)

Note that $\delta - \tau > 0$ since $\delta = 0$ or 1, and from (4.32) $\tau$ is always negative. Using the identity, $e^{\pm iv} = \cos \psi \pm i \sin \psi$, and the expanded form of $\Theta$, we may write (4.43) as follows,

$$\lambda_{SISL} = \{C - iS\} \times \{\hat{\lambda}_R \pm i\hat{\lambda}_I\},$$  \hspace{1cm} (4.44)

where,
\[ C = \sum C_r \cos[k(P^* + r)\Delta x] \]
\[ S = \sum C_r \sin[k(P^* + r)\Delta x] \]
\[ \hat{\lambda}_R = \{\delta - (\Delta t)^2\Lambda^2y^2 - \tau\}/\{\delta + (\Delta t)^2\Lambda^2\tau - \tau\} \]
\[ \hat{\lambda}_I = \{\Delta t\Lambda^\sqrt{\delta - \tau}/\{\delta + (\Delta t)^2\Lambda^2\tau - \tau\} \}
\]
\[ (4.45) \]

Therefore the amplification factor \( |\hat{\lambda}_{SISL}| = (C^2 + S^2)^{1/2}\sqrt{\lambda_R^2 + \lambda_I^2} \) may be expressed in the form,
\[ |\hat{\lambda}_{SISL}| = \left( C^2 + S^2 \right)^{1/2} \sqrt{\frac{\delta - (\Delta t)^2\Lambda^2y^2 - \tau}{\delta + (\Delta t)^2\Lambda^2\tau - \tau}}. \]
\[ (4.46) \]

To complete the solution we express \( \lambda \) in polar notation,
\[ \hat{\lambda}_{SISL} = |\hat{\lambda}_{SISL}|e^{i\omega_{SISL} \Delta t}. \]
\[ (4.47) \]

Combining (4.47) and (4.44) yields,
\[ \omega_{SISL} = -(1/\Delta t) \tan^{-1} \left( \frac{C\hat{\lambda}_I - S\hat{\lambda}_R}{C\hat{\lambda}_R + S\hat{\lambda}_I} \right) \quad \text{or} \quad \omega_{SISL} = -(1/\Delta t) \tan^{-1} \left( \frac{(\hat{\lambda}_I/\hat{\lambda}_R) - (S/C)}{1 + (S/C)(\hat{\lambda}_I/\hat{\lambda}_R)} \right) \]
\[ (4.48) \]

Following Semazzi et al. (1996), the right hand side of (4.48) may be expressed in the form,
\[ \omega_{SISL} = -(1/\Delta t) \tan^{-1} \left( \frac{\tan F - \tan G}{1 + \tan F \tan G} \right). \]
\[ (4.49) \]

where,
\[ \tan G = \frac{\sum C_r \sin[k(P^* + r)\Delta x]}{\sum C_r \cos[k(P^* + r)\Delta x]} \quad \text{and} \]
\[ \tan F = \frac{\hat{\lambda}_I}{\hat{\lambda}_R} \]
\[ (4.50) \]

Using the identity,
\[ \tan(x - y) = \tan x - \tan y \]
\[ 1 + \tan x \tan y, \]
\[ (4.51) \]

we may write \( \omega_{SISL} \) in the form, \( \omega_{SISL} = -(1/\Delta t) \tan^{-1}(\tan[F - G]) \) which is equivalent to \( \omega_{SISL} = -(1/\Delta t)(F - G) \). Therefore, the solution for \( \omega_{SISL} \) that we are seeking for (4.1) through (4.4) is,
\[ \omega_{SISL} = (1/\Delta t) \tan^{-1} \left[ \frac{\sum r C_r \sin[k(P^* + r)\Delta x]}{\sum r C_r \cos[k(P^* + r)\Delta x]} \right] \]
\[ \pm \tan^{-1} \left[ \frac{\Delta t\Lambda \sqrt{\delta - \tau}}{\{\delta + (\Delta t)^2\Lambda^2y^2\}} \right]. \]
\[ (4.52) \]

The complete solution may now be expressed in terms of (4.21) as follows,
\[ \{\bar{u}, w\} = \left( C^2 + S^2 \right)^{1/2} \sqrt{\frac{\delta - (\Delta t)^2\Lambda^2y^2 - \tau}{\delta + (\Delta t)^2\Lambda^2\tau - \tau}} \]
\[ \times \left\{ \{\bar{u}, w\}^{(i/-\Lambda x + K(m-iq)\Delta x - N_{cmax})} \right\}^N \]
\[ \times \left\{ \{\bar{u}, w\}^{(i/-\Lambda x + K(m-iq)\Delta x - N_{cmax})} \right\} (4.53) \]

Comparison of Fig. 3 and Fig. 7 confirms that the physical mode solutions based on both the first-order and second-order trajectory uncentering schemes are in close agreement.

b. Stability analysis

Although previous studies have demonstrated the unconditional stability of the semi-Lagrangian numerical integration method (Gravel et al. 1993), it is instructive to show that the particular numerical formulation we have adopted in this study is also unconditionally stable. The stability analysis of the full quartic problem (4.37) is rather complicated. Instead, we consider a slightly simplified problem, where we adopt the first-order trajectory uncentering. Thus, we examine the stability for (4.46). For the first order uncentering scheme the trajectory coefficient is defined over the range, \( 0 \leq \nu \leq 1 \). We seek to determine which of the following three stability conditions our scheme satisfies: (i) \( |\hat{\lambda}_{SISL}| = 1 \) neutral stability is assured, (ii) if \( |\hat{\lambda}_{SISL}| < 1 \) the solution is also unconditionally stable but damped with time, and (iii) if \( |\hat{\lambda}_{SISL}| > 1 \), the solution is unstable and will grow with time. We find that (4.46) reduces to the following expression,
\[ |\hat{\lambda}_{SISL}| = (C^2 + S^2)^{1/2} \sqrt{1 - \frac{(2x - 1)(\Lambda\Delta t)^2}{\{\delta + (\Delta t)^2\Lambda^2y^2\}}} \]
\[ (4.54) \]

and it follows that,
\[ |\hat{\lambda}_{SISL}| \leq (C^2 + S^2)^{1/2}, \]
\[ (4.55) \]
since \( \beta = b/\sqrt{2} \) because \( 1/2 \) \( \beta \leq 1 \). We have also used the fact that \( \tau \) in (4.32) is always negative, therefore \( \delta - \tau \) is always positive since \( \delta = 0 \) (hydrostatic) or \( \delta = 1 \) (nonhydrostatic). In other words, the scheme is unconditionally stable. The term \( (C^2 + S^2)^{1/2} \) is less than unity (Robert 1981 and 1982), and it is associated with upstream damping due to interpolation. For the fully centered scheme where, \( \alpha = \beta = 1/2 \), the only damping is due to upstream interpolation, since the second term under the square root operator in (4.54) vanishes.

c. Asymptotic analysis

In this section we are interested in examining the sensitivity of the first order trajectory uncentering scheme solutions in (4.53) on the timestep. Following Semazzi et al. (1996), for simplicity we assume motionless background flow thus, \( \Delta \beta = 0 \). In doing so we can therefore focus on the simplified forms of the analytic and numerical solutions for the frequency, which are summarized in Table 2. It is convenient to express (4.32) in the form,

\[
\tau = - \left\{ \left[ \frac{m^2 \sin(m\Delta z)}{(m\Delta z)} \right]^2 \cosh^2(q\Delta z) \right. \\
\left. + \left[ q^2 \left( \frac{\sin(q\Delta z)}{(q\Delta z)} \right)^2 \cos^2(m\Delta z) \right] \right\} / k^2 \left( \frac{\sin k\Delta x}{k\Delta x} \right)^2.
\]

In order to isolate the temporal truncation error associated with the SISL scheme we consider the limit as \( \Delta z \to 0 \) and \( \Delta x \to 0 \) (i.e. no spatial truncation errors). Thus, the term \( \delta - \tau \) in the equation for \( \omega_{SISL} \) in Table 2 may be stated as follows,

\[
\delta - \tau = \frac{\delta k^2 + m^2 + q^2}{k^2}.
\]

Use of (4.57) together with the definition of \( \omega_a \) in (A3; neglecting the Lagrangian advection term, \( \Delta \beta \)), we express the numerical frequency \( \omega_{SISL} \) in the form,

\[
\omega_{SISL} = \pm (1/\Delta t) \tan^{-1} \left[ \frac{\Delta \omega_a}{1 - (\Delta t)^2 \Delta \beta \omega_a} \right].
\]
In arriving at (4.58) we have neglected the first term of \( \omega_{\text{SISL}} \) in (4.52), which corresponds to Lagrangian advection. Applying Taylor series expansion to the term in the square brackets of (4.58) we obtain,

\[
\omega_{\text{SISL}} = \pm (1/\Delta t) \tan^{-1} \left[ \Delta t \omega_a (1 + \Delta t^2 \beta \omega_a + O(\Delta t^4) + \cdots) \right].
\]

(4.59)

Further application of Taylor series expansion to the tangent term we get,

\[
\omega_{\text{SISL}} = \pm (1/\Delta t) \left\{ (\Delta t \omega_a [1 + \Delta t^2 \beta \omega_a + O(\Delta t^4)]) - \frac{(\Delta t \omega_a [1 + \Delta t^2 \beta \omega_a + O(\Delta t^4)]^3}{3} + \cdots \right\}.
\]

(4.60)

Noting that \( a \beta = [1 - (\xi_f^2)}/4 \) and retaining up to \( O(\Delta t)^2 \) terms yields,

\[
\frac{\omega_{\text{SISL}}}{\omega_a} \approx 1 - (\Delta t)^2 \omega_a^2 \{1 + 3(\xi_f^2)^2)/12 \}.
\]

(4.61)

Since, \( \omega_a(\text{nonhydrostatic}) < \omega_a(\text{hydrostatic}) \), (4.61) is consistent with our earlier observation based on Fig. 3 that \( \omega_{\text{SISL}}(\text{nonhydrostatic}) \) is more accurately simulated than \( \omega_{\text{SISL}}(\text{hydrostatic}) \). Inspection of the variation of \( \omega_{\text{SISL}} \) as a function of the size of the time step, and the corresponding graph for the asymptotic solution (not displayed), indicates that the two are in close agreement. We further note that the error increases with the uncentering parameter and it is minimal for the fully centered scheme (\( \xi_f = 0 \)).

6. Conclusions

We have investigated the accuracy of the semi-implicit semi-Lagrangian (SISL) method in simulating internal gravity wave (IGW) motion in the atmosphere. The study focuses on the relative accuracy of the scheme in simulating hydrostatic and nonhydrostatic IGW motion. The analysis is based on SISL solutions of a linearized model, and a Global Circulation Model-Dynamic Core (GCM-DC).

The GCM-DC model is based on the two time level SISL numerical formulation. It has been adapted from an earlier version of the model described in Semazzi et al. (1995) and Qian et al. (1998a). The main modification is the replacement of the uniform resolution grid with a stretched variable grid mesh to facilitate the simulation of meso-scale IGW motion. The GCM-DC simulations consist of a pair of identical twin experiments. Starting from zonal initial motion, stationary IGW disturbances are generated by introducing an isolated meso-scale Witch of Agnesi mountain, with half-width of 2 km (wavelength \( \approx 4 \) km), at the bottom of the atmosphere. A binary switch in the model has been used to suppress, or retain the terms in the governing equations responsible for nonhydrostatic dynamics.

In the control GCM-DC experiments a modest Courant number of 0.5 was used. The nonhydrostatic version of the model produces the familiar meso-scale gravity wave train disturbance anchored to the isolated mountain. The wave has a distinct tilt, away from the vertical, that is consistent with the classical analytical solutions (Smith 1980). When the nonhydrostatic terms of the model are suppressed, the axis of the resulting gravity wave train rests nearly perpendicular to the mountain top, thus consistent with the classical theory of meso-scale gravity wave dynamics. Increasing the time step from 10 s to 60 s, which corresponds to a Courant number of 3, results in solutions which are stable for both the hydrostatic and nonhydrostatic versions of the model. The axis of the nonhydrostatic gravity wave train is tilted away from the vertical direction.

<table>
<thead>
<tr>
<th>Table 2. Summary of solutions for the linear model</th>
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<tbody>
<tr>
<td><strong>Analytical Form</strong></td>
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<tr>
<td><strong>Frequency</strong></td>
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<tr>
<td><strong>Amplitude</strong></td>
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thus in close agreement with the control run. However, in the case of the hydrostatic version of the model, the increase in the Courant number results in large, and potentially unacceptable angular shift in the vertical tilt, that is not present in the control run when the Courant number is significantly smaller.

To further investigate the peculiar behavior of the GCM-DC solutions at large Courant number, we have also examined the numerical solutions of a linearized SISL model for which the analytic solutions are also derived to assess the performance of the numerical scheme. At the horizontal scale of $L_x = 20 \text{ km}$, which is representative of long wavelength meso-scale motion, the distinction between hydrostatic and nonhydrostatic motion is negligible. The SISL numerical scheme reproduces the corresponding analytic solutions, with no significant loss of accuracy for Courant number of up to 3. At $L_x = 5 \text{ km}$, consistent with short wavelength meso-scale motion, and comparable to the mountain wavelength $\approx 4 \text{ km}$ in the GCM-DC, the nonhydrostatic effects are clearly significant. The solutions of the linear model are qualitatively consistent with the GCM-DC results, in that the IGW motion is more accurately simulated in the nonhydrostatic formulation of the model than in the corresponding hydrostatic version. As $\Delta t \to \infty$ the frequency of the computational SISL mode asymptotically approaches the value of the frequency for the corresponding physical SISL mode. Since the amplitude for the computational SISL mode is directly proportional to the size of time step, the results suggest that the resulting amplification may have positive contribution of offsetting some of the damping associated with the SISL physical mode, particularly at large time steps.

With increasing $\Delta t$ for Courant number near or greater than one, the temporal truncation error in the solution of the physical mode increases rapidly and becomes dominant over the spatial truncation error. Error growth relative to the corresponding analytical solution is proportional to the squares of the time step and the analytical wave frequency. It is apparent that higher-order ($2^{nd}$-order and higher) SISL scheme may be required to overcome the potentially unacceptable loss of accuracy at high Courant numbers.

Appendix A

Analytical solutions for the linear model

As in section 4 we assume Boussinesq conditions and harmonic/exponential dependency of the form,

$$\begin{align*}
\{u, w\} &= \begin{pmatrix} \hat{u} e^{ikx+(m-iq)z-\alpha_0 t} \\
\hat{w} e^{ikx+(m+iq)z-\alpha_0 t} \end{pmatrix} \\
\{\rho, \phi\} &= \begin{pmatrix} \hat{\rho} e^{ikx+(m+iq)z-\alpha_0 t} \\
\hat{\phi} e^{ikx+(m-iq)z-\alpha_0 t} \end{pmatrix}
\end{align*}$$

(A1)

Following Nappo (2002), the linear system of governing equations (4.1) through (4.4) may be expressed in matrix form as follows,

$$\begin{pmatrix}
(c_{0a} - \alpha k) & 0 & 0 & \frac{1}{2} he^{-2qz} \\
0 & i(c_{0a} - \alpha k)q - \frac{3}{4} e^{-2qz} & \frac{3}{2} (im - q)e^{-2qz} \\
0 & \frac{2}{3} qz & i(c_{0a} - \alpha k)e^{-2qz} & 0 \\
i k & (im + q) & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\hat{u} \\
\hat{w} \\
\hat{\rho} \\
\hat{\phi}
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}$$

(A2)

We note that the exponential term drops out. For non-trivial solutions, the matrix on the left hand side must be singular, thus its determinant identically equal to zero. The result is a dispersion formula for the analytical frequency, which may be stated as follows,

$$\omega_a = \alpha_k \pm \frac{k \Lambda}{\left( m^2 + q^2 + k^2 \delta \right)^{1/2}}$$

(A3)

Here, (A3) is the analytical analogue of the corresponding numerical form in (4.52) arising from semi-Lagrangian discretization of (4.1) through (4.4). The term involving $(q)$ is usually taken to be small below the tropopause and ignored (Nappo 2002). Note that according to (A1) the analytical amplitude $|\omega_a| = 1$, and the corresponding numerical form is given by (4.46).

Appendix B

Roots for the fourth order polynomial problem

The roots for the fourth-order polynomial problem in (4.37) may be found in many standard mathematical formula books (e.g., Rade and Westergren 1989) and expressed as follows,

$$\chi_{\text{SISL}(1,2)} = -\frac{1}{4} \frac{\mu_3}{\mu_4} + \frac{1}{2} \sqrt{\frac{1}{2} \delta}$$

(B1)
\[ \lambda_{SISL(3,4)} = -\frac{1}{4} \mu_4 \frac{1}{\mu_4} - \frac{1}{2} \hat{R} + \frac{1}{2} i \hat{E} \]  

(B2)

The linear model quartic solutions in (B1) and (B2) are discussed in section 5.1. The functions (\( \hat{R} \)), (\( \hat{D} \)), and (\( \hat{E} \)) are given by,

\[ \hat{R} = \sqrt{\frac{1}{4} 4^{3/2} \mu_4^3 - \mu_2 \mu_4 + \hat{N}} \]  

(B3)

\[ \hat{D} = -\frac{1}{4} \hat{R} \left(4 \mu_2 \mu_4 \mu_4^2 - 8 \mu_1 \mu_4 - \mu_3 \mu_4^2 \right) - \frac{3}{4} \mu_3 \mu_4^2 + \hat{R}^2 + 2 \mu_2 \mu_4 \]  

if \( \hat{R} \neq 0 \)

\[ \hat{E} = \frac{1}{4} \hat{R} \left(4 \mu_2 \mu_4 \mu_4^2 - 8 \mu_1 \mu_4 - \mu_3 \mu_4^2 \right) - \frac{3}{4} \mu_3 \mu_4^2 + \hat{R}^2 + 2 \mu_2 \mu_4 \]  

if \( \hat{R} \neq 0 \)

\[ \hat{N} = \sqrt{\frac{3}{2} - \frac{\sqrt{\hat{Q}^2 + 4 \hat{P}^3}}{2} - \frac{\sqrt{\hat{Q}^2 + 4 \hat{P}^3}}{2}}. \]  

(B7)

where, \( \hat{P} \) and \( \hat{Q} \) are functions of the primary coefficients in (4.38). They are defined as follows,

\[ \hat{P} = \frac{3(\mu_3 \mu_4 \mu_4^2 - 4 \mu_0 \mu_4) - \mu_2^2}{9} \]  

(B8)

and,

\[ \hat{Q} = \left( -2 \mu_3 \mu_4^2 + 9(\mu_2 \mu_4)(\mu_3 \mu_4^2 - 4 \mu_0 \mu_4) \right) + 27 (4 \mu_2 \mu_0 \mu_4^2 - \mu_4 \mu_3 \mu_4^2 \mu_4^2) \) \]  

(B9)

\[ \lambda = \text{time dependent factor of } u, v, \phi. \]

where, \( u, v, \phi \):

perturbations from \( \bar{u}, \bar{v}, \bar{w}, \bar{\rho}, \bar{\phi} \).

\[ \bar{u} = 0, \bar{v}, \phi; \]

amplitude corresponding to \( u, v, \phi \).

\[ x, y, z, t; \]

orthogonal coordinates in the zonal, meridional, and time dimensions, respectively.

\[ \eta; \]

first order uncentering parameter.

\[ \kappa; \]

second order uncentering parameter.

<table>
<thead>
<tr>
<th>Earth’s gravitational acceleration (( = 9.8 \text{ m s}^{-2} ))</th>
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<tbody>
<tr>
<td>Temperature of isothermal basic state</td>
</tr>
</tbody>
</table>

\[ \Delta x, \Delta y, \Delta z, \Delta t; \]

grid spacing in the zonal direction, the meridional direction, vertical direction, and time, respectively.

\[ I, K, N; \]

grid index in the zonal direction, vertical direction and time, respectively.

\( t \):

designates departure point at \( t \)

\( \Delta t \):

designates departure point at \( t - \Delta t \).

\( \lambda \):

time dependent factor of \( u, v, \phi \).

\( g \):

acceleration due to gravity

\( (k) \):

wave number, \( (2\pi/\text{zonal wave length}) \).

\( \mu \):

wave number, \( (2\pi/\text{vertical wave length}) \).

\( \Delta x \):

time between departure point at \( t \) & arrival point expressed in terms of the number of grid spacings.

\( p \):

integer part of \( \Delta x \)

\( q \):

fractional part of \( \Delta x \)

\( \Delta x^* \):

time between departure point at \( t - \Delta t \) & arrival point expressed in terms of the number of grid spacings.

\( p^* \):

integer part of \( \Delta x^* \)
\( \dot{z}^{**} \): fractional part of \( z^{**} \)

\( SI \): abbreviation for semi-implicit

\( SISL \): abbreviation for semi-implicit semi-Lagrangian

\( T_b \): isothermal temperature of the basic state

\( \omega_a \): wave frequency corresponding for the analytical solution.

\( \omega_{SISL} \): numerical wave frequency for the linear semi-Lagrangian model

\( \Lambda^2 \): Brunt-Vaisala frequency of the basic state

\( \delta \): Switch for hydrostatic approximation

\( \Gamma \): Switch for Boussinesq approximation

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References


