Potential Vorticity Dynamics of Tropical Cyclone

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Outlines

- Filamentation Time Diagnosis with potential vorticity (PV) dynamics for synaptic scale weather pattern and tropical cyclone.

- Available Energy in an Axisymmetric Vortex
  PV is highly related to heating efficiency.

- Ekman-Taylor Layer under a Circular Vortex
  Ekman pumping is sensitive to PV distribution.

Collaboration with Prof. Wayne H. Schubert,
Colorado State University, Fort Collins, Colorado
Filamentation Time Diagnosis

Conserves the angular impulse

\[ \iint (x^2 + y^2) \zeta \, dx \, dy \]

Melander et al. 1986
Derivation of Filamentation Time

Based on deriving vorticity gradient. From barotropic vorticity equation:

\[
\frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} = \nu \nabla^2 \zeta
\]  

(1)

If neglecting diffusion \( \nu = 0 \), and computing \( \partial(1)/\partial x \pm i \partial(1)/\partial y \),
We could obtain the time dependent equation of vorticity gradient:

\[
\frac{D_2}{Dt} \begin{pmatrix} \frac{\partial \zeta}{\partial x} + i \frac{\partial \zeta}{\partial y} \\ \frac{\partial \zeta}{\partial x} - i \frac{\partial \zeta}{\partial y} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} i \zeta & -(S_n + i S_s) \\ -(S_n - i S_s) & -i \zeta \end{pmatrix} \begin{pmatrix} \frac{\partial \zeta}{\partial x} + i \frac{\partial \zeta}{\partial y} \\ \frac{\partial \zeta}{\partial x} - i \frac{\partial \zeta}{\partial y} \end{pmatrix}
\]  

(2)

\begin{align*}
\begin{cases}
S_n = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} & \text{Stretching deformation} \\
S_s = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} & \text{Shearing deformation}
\end{cases}
\end{align*}

\[\vec{\nabla} \zeta(t) \propto \exp(\lambda t)\]

\[\lambda = \pm \frac{1}{2} \sqrt{Q} = \pm \frac{1}{2} \sqrt{S_n^2 + S_s^2 - \zeta^2}\]

Rozoff et al. (2006)
Deformation Field in Cartesian Coordinates

Stretching Deformation

Shearing Deformation

Bluestein (1992)
\begin{equation}
\mathbf{\nabla} \zeta(t) \propto \exp \left[ \pm \frac{1}{2} \sqrt{S_n^2 + S_s^2 - \zeta^2} \ t \right] \quad \text{Vorticity gradient will be stretched}
\end{equation}

\begin{equation}
\mathbf{\nabla} \zeta(t) \propto \exp \left[ \pm \frac{1}{2} i \sqrt{\zeta^2 - S_n^2 - S_s^2} \ t \right] \quad \text{Vortex is stable}
\end{equation}

In strain dominated region \( (S_n^2 + S_s^2 > \zeta^2 \to Q > 0) \)

\[ \mathbf{\nabla} \zeta(t) \propto \exp \left[ \pm \frac{1}{2} \sqrt{S_n^2 + S_s^2 - \zeta^2} \ t \right] \]

Define

\[ \tau_{fil} = 2 \sqrt{S_n^2 + S_s^2 - \zeta^2}^{-1} \]

Rozoff et al. (2006)
Filamentation Time

\[ \tau_{fil} = 1 / \max \{ \lambda_i \} \quad \text{for} \quad \lambda_i \in \mathbb{R}, \]

- **Okubo-Weiss type** (Weiss, 1991)

\[ \tau_{fil} = \frac{2}{\sqrt{S_n^2 + S_s^2 - \zeta^2}} \]

- **Hua-Klein type** (Hua and Klein, 1998)

considering the acceleration gradient tensor

\[ \tau_{fil} = \frac{1}{\sqrt{\frac{1}{4}(S_n^2 + S_s^2 - \zeta^2) + \frac{1}{2}\sqrt{\dot{S}_n^2 + \dot{S}_s^2 - \dot{\zeta}^2}}} \]

Where \( S_n \): Stretching Deformation. \( S_s \): Shearing Deformation.

\[
\dot{S}_n = \left( \frac{\partial}{\partial t} + \bar{u} \cdot \nabla \right) \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) \\
\dot{S}_s = \left( \frac{\partial}{\partial t} + \bar{u} \cdot \nabla \right) \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \\
\dot{\zeta} = \left( \frac{\partial}{\partial t} + \bar{u} \cdot \nabla \right) \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)
\]
Barotropic instability with Okubo-Weiss and Hua-Klein filamentation time diagnosis
Nondimensional moat width v.s. nondimensional filamentation moat width

Cat 4

Cat 3

Cat 5

Kuo et al. 2009
The Application of Filamentation Time

- The moat formation of typhoon (Rozoff et al., 2006) or formation of inner spiral rainbands (Wang 2008)

- Synoptic scale trough thinning or broadening and the formation of cutoff low.

Two paradigms of baroclinic-wave life-cycle behaviour

Thornicroft, Hoskins, and McIntyre (1993, QJRMS)
PV contour on isentropic surface

\[ \zeta = \nabla^2 \psi \]

LC1 Trough Thinning

LC2 Trough Broadening

Thornicroft et al. (1993)
Spherical Filamentation Time on Isentropic Surface

Inviscid, adiabatic, quasi-static flow in the atmosphere obeys the material conservation relation

\[
\frac{DP}{Dt} = 0 \quad (A)
\]

where

\[
P = \left( -\frac{1}{g \theta} \right)^{-1} \left( 2\Omega \sin \phi + \frac{\partial v}{a \cos \phi \partial \lambda} - \frac{\partial (u \cos \phi)}{a \cos \phi \partial \phi} \right)_\theta
\]

is the potential vorticity and

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{a \cos \phi \partial \lambda} + v \frac{\partial}{a \partial \phi}
\]

is the material derivative.
Spherical Filamentation Time on Isentropic Surface

To understand the evolution of the PV gradient, we first differentiate (A) with respect to $\lambda$ and $\phi$ to obtain

$$\frac{D}{Dt} \left( \frac{\partial P}{a \cos \phi \partial \lambda} \right) + \left( \frac{\partial }{\partial \phi} \left( \frac{\partial u}{a \cos \phi \partial \lambda} - \frac{v \tan \phi}{a} \frac{\partial v}{a \cos \phi \partial \lambda} \right) \right) \left( \frac{\partial P}{a \cos \phi \partial \lambda} \right) = 0$$

(B)

<table>
<thead>
<tr>
<th>Isentropic divergence</th>
<th>Isentropic stretching deformation</th>
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<tbody>
<tr>
<td>$\delta = \frac{\partial u}{a \cos \phi \partial \lambda} + \frac{\partial (v \cos \phi)}{a \cos \phi \partial \phi}$</td>
<td>$S_n = \frac{\partial u}{a \cos \phi \partial \lambda} - \frac{\cos \phi}{a \partial \phi} \left( \frac{v}{\cos \phi} \right)$</td>
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<th>Isentropic relative vorticity</th>
<th>Isentropic shearing deformation</th>
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</tr>
</tbody>
</table>
Spherical Filamentation Time on Isentropic Surface

\[
\mathcal{V} = \begin{pmatrix}
\frac{1}{2}(\delta + S_n) & \frac{1}{2} \left( S_s + \zeta - \frac{2u \tan \phi}{a} \right) \\
\frac{1}{2} \left( S_s - \zeta + \frac{2u \tan \phi}{a} \right) & \frac{1}{2}(\delta - S_n)
\end{pmatrix}
\]

The eigenvalues of \( \mathcal{V} \) are

\[
\mu = \frac{1}{2} \left\{ \delta \pm \left[ S_n^2 + S_s^2 - \left( \zeta - \frac{2u \tan \phi}{a} \right)^2 \right]^{1/2} \right\}
\]
Spherical Filamentation Time on Isentropic Surface

The solutions of (B) are approximately linear combinations of

\[
\exp\left\{\frac{1}{2} \left[ \delta \pm i \left( \left( \zeta - \frac{2u \tan \phi}{a} \right)^2 - S_n^2 - S_s^2 \right)^{1/2} \right] t \right\}
\]

if \( \left( \zeta - \frac{2u \tan \phi}{a} \right)^2 > S_n^2 + S_s^2 \) (rotation dominated),

or linear combinations of

\[
\exp\left\{\frac{1}{2} \left[ \delta + \left( S_n^2 + S_s^2 - \left( \zeta - \frac{2u \tan \phi}{a} \right)^2 \right)^{1/2} \right] t \right\}
\]

(C)

if \( \left( \zeta - \frac{2u \tan \phi}{a} \right)^2 < S_n^2 + S_s^2 \) (strain dominated).
Spherical Filamentation Time on Isentropic Surface

For the strain dominated case, we can define the filamentation time as the e-folding time of the exponential solution (C), i.e.,

\[
\tau_{\text{fil}} = 2 \left\{ \delta + \left[ S_n^2 + S_s^2 - \left( \frac{\zeta - 2u \tan \phi}{a} \right)^2 \right]^{1/2} \right\}^{-1}
\]

for \( S_n^2 + S_s^2 > \left( \frac{\zeta - 2u \tan \phi}{a} \right)^2 \).

(D)

Define the area with short filamentation time as filamentation zone.
Synoptic Trough Analysis

**Strong Baroclinicity**

Event Duration Time:
1990 Jan 28 00 UTC ~ 1990 Jan 29 12UTC

Event Features: Both trough thinning and broadening pattern can be seen at the same time on the same weather plot. The cutoff low is formed at 1990 Jan 29 00 UTC.

Data Source: ECMWF basic data

Resolution: 2.5 * 2.5
310 K Isentropic Potential Vorticity

Shading: 1 to 2 PVU (dynamic tropopause)

James, 1994
Contour: PV on isentropic surface
Shading: Filamentation time on isentropic surface
Synoptic Trough Analysis

**Moderate Baroclinicity**
during the Mei-Yu season

Event Features: A typical and obvious cutoff low formed near Taiwan area during the Mei-Yu season. The cutoff low is formed at 1998 May 05 12 UTC.

Analysis Time:
1998 May 04 00 UTC ~ 1998 May 05 12 UTC (every 12 hrs)

Data Source: ECADV

Resolution: 0.5 * 0.5
Contour: PV on isentropic surface
Shading: Filamentation time on isentropic surface
Synoptic Trough Analysis

At Tropics

Event Features: A TUTT cell cut from a TUTT. The TUTT cell is formed at 2005 June 04 12 UTC.

Analysis Time:
2005 June 02 12 UTC ~ 2005 June 04 12 UTC (every 12 hrs)

Data Source: ECADV

Resolution: 1.125 * 1.125
Contour: PV on isentropic surface
Shading: Filamentation time on isentropic surface
Discussion

- The importance of curvature terms and divergence term in the filamentation time calculation
- The effects of stretching and shearing deformation
Exact

\[ \mu = \frac{1}{2} \left\{ \delta \pm \left[ S_n^2 + S_s^2 - \left( \zeta - \frac{2u \tan \phi}{a} \right)^2 \right]^{1/2} \right\} \]

Without curvature terms

\[ \mu = \frac{1}{2} \left\{ \left( \frac{\partial u}{a \cos \phi \partial \lambda} + \frac{\partial v}{a \partial \phi} \right) \pm \left[ \left( \frac{\partial u}{a \cos \phi \partial \lambda} - \frac{\partial v}{a \partial \phi} \right)^2 \right. \\
+ \left( \frac{\partial v}{a \cos \phi \partial \lambda} + \frac{\partial u}{a \partial \phi} \right)^2 - \left( \frac{\partial v}{a \cos \phi \partial \lambda} - \frac{\partial u}{a \partial \phi} \right)^2 \right]^{1/2} \right\} \]

Without divergence term

\[ \mu = \pm \frac{1}{2} \left\{ \left[ \left( \frac{\partial u}{a \cos \phi \partial \lambda} - \frac{\partial v}{a \partial \phi} - \frac{u \tan \phi}{a} \right)^2 \right. \\
+ \left( \frac{\partial v}{a \cos \phi \partial \lambda} + \frac{\partial u}{a \partial \phi} + \frac{u \tan \phi}{a} \right)^2 - \left( \frac{\partial v}{a \cos \phi \partial \lambda} - \frac{\partial u}{a \partial \phi} - \frac{u \tan \phi}{a} \right)^2 \right]^{1/2} \right\} \]
In the cutoff region, for PV larger than 1 PVU, the error is 13.5%. 

\[
error(\%) = \left| \frac{\tau_{fil}^{div} - \tau_{fil}^{nodiv}}{\tau_{fil}^{div}} \right| \times 100
\]
In the cutoff region, for PV larger than 2 PVU, the error is 11.6%.

\[
\text{error(\%)} = \left| \frac{(\tau_{\text{fil}})_{\text{div}} - (\tau_{\text{fil}})_{\text{nodiv}}}{(\tau_{\text{fil}})_{\text{div}}} \right| \times 100
\]
Isentropic PV and Filamentation time

Stretching Deformation

1998 May 04 12Z

(\text{\textit{Units:}}} 10^{-8} (s^{-2})\)
Isentropic PV and Filamentation time

2005 Jun 03 12Z

Stretching Deformation

Shearing Deformation

$\times 10^{-9}$

$(s^{-2})$
Summary

- We generalize the concept from 2D turbulence to the analysis of synoptic scale phenomenon with the filamentation time in spherical isentropic coordinates.

- Filamentation time gives a quantitative diagnosis for straining out process. (the formation of band like structure)

- Filamentation time is generally longer in the tropics than in midlatitudes.

- Because the curvature terms are only significant at high latitudes, they can be ignored for simplicity, while the isentropic divergence term must be retained for quantitative calculation.
In addition to the PV information, the filamentation time diagnosis contains the effects of divergence, stretching and shearing deformations.

A combination of cross-PV-contour flow analysis on isentropic surfaces (Thorncroft et al. 1993) with the filamentation time analysis gives a more complete description of the dynamics.
Available Energy
in an Axisymmetric Vortex
The fact that the Available Potential Energy (APE) is an upper bound on the amount of kinetic energy that can be generated by any circulation however is both a strength and a weakness.

- It is a strength because the concept is completely general. It is a weakness because it is possible that no dynamically realizable circulation can extract all of the available energy. (Randall and Wang 1992)

Heating efficiency
Hack and Schubert (1986)
Lorenz (1955) considered the total potential energy of an adiabatic frictionless fluid.

\[ \text{APE} = H_{gs} - H_{rf} \]

Enthalpy

\[
\bar{A} = \frac{R}{2gp_{00}^{\kappa}} \int_0^{p_s} \frac{\bar{\theta}^2}{(p^{1/\kappa})(-\frac{\partial \bar{\theta}}{\partial p})} \cdot \frac{\theta'}{\bar{\theta}} dp
\]
Literature Review

- The axisymmetric fluid motions with the form of small departures from a steady axisymmetric reference is considered in van Mieghem (1956).
- Andrews (1981, hereafter A81) proposed a theory of positive definite available potential energy for a compressible, stratified, nonhydrostatic fluid.
A Unified Theory of Available Potential Energy

- Underlying Hamiltonian structure.
- APE is just the non-kinetic part of the pseudoenergy.
- A special feature of non-canonical representation is that they possess a class of invariants known as Casimir invariants.

- Energy of acoustic waves
- Energy of internal gravity waves
- Lorenz’s available potential energy
- Available potential energy of a compressible, stratified fluid
- Available energy for disturbances to non-resting basic states

Shepherd 1993
Literature Review

- Andrews (2006, hereafter A06) derived an available energy for compressible fluid referred to an axisymmetric reference atmosphere through a method similar to A81.

- A06 also extends van Mieghem (1956) to finite amplitude disturbances.
\[ A = -\frac{1}{2} \left( \frac{\partial \chi}{\partial \mu_0} \right)_{\mu_0} (\mu_0 - \mu_0)^2 - \left( \frac{\partial \chi}{\partial s_0} \right)_{\mu_0} (\mu_0 - \mu_0)(s_0 - s_0) - \frac{1}{2 \rho_0 c_p} \left( \frac{\partial p_0}{\partial s_0} \right)_{\mu_0} (s_0 - s_0)^2 + \frac{1}{2 \rho_0^2 c_0^2} (p_0 - p_0)^2, \]

\[ s = c_p \ln \left( \frac{T}{T_{00}} \right) - R \ln \left( \frac{p}{p_{00}} \right) = c_v \ln \left( \frac{T}{T_{00}} \right) - R \ln \left( \frac{\rho}{\rho_{00}} \right), \]

where \( T_{00} \) is the reference temperature, 273.15K; \( p_{00} \) the reference pressure of dry air, 100kPa; and \( \rho_{00} \) the reference density of dry air at \( T_{00} \) and \( p_{00} \).

Andrews (2006)

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**The Reference State**

\[ \mathbf{u}_0 \cdot \nabla \mathbf{u}_0 + \frac{1}{\rho_0} \nabla p_0 + \nabla \Phi = 0. \]

\[ \frac{\partial \chi}{\partial s_0} \left( \frac{\partial T_0}{\partial \mu_0} \right)_{s_0} = \left( \frac{\partial T_0}{\partial \mu_0} \right)_{s_0}, \]

\[ \rho_0^{-1} \left( \frac{\partial p_0}{\partial z} \right)_R + \left( \frac{\partial \Phi}{\partial z} \right)_R = 0. \]

\[ \left( \frac{\partial \mu_0}{\partial z} \right)_R = 2 m_0 \rho_0 R^4 p_0 \left( \frac{\partial T_0}{\partial \mu_0} \right)_{s_0}. \]

Andrews (2006)
Nonlinear Response of Atmospheric Vortices to Heating by Organized Cumulus Convection

Motivation

Fig. 3. Radius of maximum wind and maximum low-level tangential wind as a function of time for the invariantly forced balance system. Dotted lines correspond to a maximum forcing at 400 mb, solid lines correspond to maximum forcing at 500 mb, and dashed lines correspond to maximum forcing at 600 mb.

Hack and Schubert 1986
\[ \frac{dP}{dt} = H - C, \]  \hspace{1cm} (3.1)

\[ \frac{dK}{dt} = C, \]  \hspace{1cm} (3.2)

where

\[ P = \int \int c_p T \rho r dr dz, \]  \hspace{1cm} (3.3) \text{Total potential energies}

\[ K = \int \int \frac{1}{2} v^2 \rho r dr dz, \]  \hspace{1cm} (3.4) \text{Total kinetic energies}

\[ H = \int \int Q \rho r dr dz, \]  \hspace{1cm} (3.5) \text{Total heating}

\[ C = \int \int \frac{g}{\theta_0} w \theta \rho r dr dz. \]  \hspace{1cm} (3.6) \text{Rate of conversion of total potential energy to kinetic energy}

Hack and Schubert 1986
\[ C = \int \int \frac{g}{\theta_0} w^* \theta \rho R dR dZ. \quad (3.7) \]

\[ C = -\int \int \psi^* \frac{g}{\theta_0} \frac{\partial \theta}{\partial R} R dR dZ. \quad (3.8) \]

\[ L^* \psi^* = \frac{g}{\theta_0} \frac{\partial Q}{\partial R} \]

\[ L^* \chi^* = \frac{g}{\theta_0} \frac{\partial \theta}{\partial R} \]

\[ C = -\int \int \psi^* L^* \chi^* R dR dZ = -\int \int \chi^* L^* \psi^* R dR dZ \]

Self-Adjoint property

\[ C = \int \int \eta^* Q \rho R dR dZ \quad \rightarrow \quad C = \int \int \eta Q \rho r d r d z \]

\[ \tilde{\eta} = \frac{C}{H} = \frac{\int \int \eta Q \rho r d r d z}{\int \int Q \rho r d r d z} \quad \text{System efficiency factor} \]

Hack and Schubert 1986
Heating at lower level (400 hPa) is more efficient than at upper level (600 hPa).

Small changes in the diabatic heating profile can have such a profound impact on the evolution of the large-scale flow.

Hack and Schubert 1986
The derivation of the available energy principle for a compressible, stratified, nonhydrostatic, dry atmosphere. Extend Andrews (2006) with Coriolis effect.
Using cylindrical coordinates \((r, z)\), the prognostic equations for a baroclinic circular vortex are

\[
\begin{align*}
\frac{Du}{Dt} - \left( f + \frac{v}{r} \right) v + \frac{1}{\rho} \frac{\partial p}{\partial r} &= 0, \\
\frac{Dv}{Dt} + \left( f + \frac{v}{r} \right) u &= 0, \\
\frac{Dw}{Dt} + g + \frac{1}{\rho} \frac{\partial p}{\partial z} &= 0, \\
\frac{D\rho}{Dt} + \rho \left( \frac{\partial (ru)}{r \partial r} + \frac{\partial w}{\partial z} \right) &= 0, \\
\frac{Ds}{Dt} &= \frac{Q}{T}, \\
p &= \rho RT,
\end{align*}
\]

There are 2 ways to express (3.5)

\[
\begin{align*}
\text{Internal energy form} \\
\rho c_v \frac{DT}{Dt} &= \frac{p}{\rho} \frac{D\rho}{Dt} + \rho Q \\
\text{Enthalpy form} \\
\rho c_p \frac{DT}{Dt} &= \frac{Dp}{Dt} + \rho Q \\
\text{Enthalpy} \\
c_p T &= c_v T + p/\rho
\end{align*}
\]
Enthalpy form

\[ \rho \frac{D}{Dt} \left[ \frac{1}{2}(u^2 + w^2) + \frac{\mu}{2r^2} + c_p T \right] + \left[ g z + \frac{1}{8} f^2 r^2 \right] \left( \frac{p - p_0}{\rho} \right) + \frac{\partial [ru(p - p_0)]}{\partial r} + \frac{\partial [w(p - p_0)]}{\partial z} = \rho Q. \] (3.28)

Gradient and hydrostatic equations for the reference state

\[
\left( f + \frac{v_0}{r} \right) v_0 = \frac{1}{\rho_0} \frac{\partial p_0}{\partial r}, \quad -g = \frac{1}{\rho_0} \frac{\partial p_0}{\partial z}
\]

\[ C_0 = \tilde{C}(\mu_0, s_0) \]

\[ \rho \frac{D}{Dt} \left[ \frac{1}{2}(u^2 + w^2) + A \right] + \frac{\partial [ru(p - p_0)]}{r \partial r} + \frac{\partial [w(p - p_0)]}{\partial z} = \left( 1 + \frac{\tilde{C}_s(\mu, s)}{T} \right) \rho Q. \] (3.31)

where the available energy per unit mass \( A \) is defined by

\[ A = \tilde{C}(\mu, s) - \tilde{C}(\mu_0, s_0) - (\mu - \mu_0) \tilde{C}_\mu(\mu_0, s_0) + c_p T(s, p) - c_p T(s_0, p_0) - \frac{p - p_0}{\rho}. \] (3.32)

where \( \mu_0 = m_0^2 \) and \( m_0 = rv_0 + \frac{1}{2} f r^2 \).

\( m_0 \) is the angular momentum in cylindrical coordinates.
Assume ideal gas law

\[ T(s, p) = T_0 e^{s/c_p} \left( \frac{p}{p_0} \right)^\kappa, \quad \frac{1}{\rho(s, p)} = \frac{RT_0}{p_0} e^{s/c_p} \left( \frac{p}{p_0} \right)^{\kappa-1} \]

\[
A = \tilde{C}(\mu, s) - \tilde{C}(\mu_0, s_0) + \frac{m^2 - m_0^2}{2r^2} + (s - s_0)T(s_0, p_0)
+ c_p T_0 e^{s/c_p} \left( \frac{p_0}{p_0} \right)^\kappa \left[ (1 - \kappa) \left( \frac{p}{p_0} \right)^\kappa + \kappa \left( \frac{p}{p_0} \right)^{\kappa-1} - 1 \right]
+ c_p T_0 e^{s_0/c_p} \left( \frac{p_0}{p_0} \right)^\kappa \left( e^{(s-s_0)/c_p} - 1 - \frac{s - s_0}{c_p} \right).
\]

Integrating (3.31) over the whole domain, we obtain

\[
\frac{d}{dt} \iint [\frac{1}{2}(u^2 + w^2) + A] \rho r \, dr \, dz = \iint \left( \frac{T - \hat{T}}{T} \right) Q \rho r \, dr \, dz, \tag{3.42}
\]

where

\[ \hat{T} = T(s_0, p_0) \frac{\tilde{C}_s(\mu, s)}{\tilde{C}_s(\mu_0, s_0)}. \]

**Absolute angular momentum information
Entropy

Note that \( (T - \hat{T})/T \), known as the “Carnot factor” (Emanuel 1986, 1988), determines the “energetic efficiency” of the diabatic heating.
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<th>Conditions</th>
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<td>Lorenz (1955)</td>
<td>[ A = \frac{R}{2g\rho'<em>{0}} \int</em>{0}^{p_s} \frac{\theta'}{(p^{1-\kappa})(-\frac{\partial \theta}{\partial p})} \cdot \left(\frac{\partial'}{\theta}\right) dp ]</td>
<td>Hydrostatic reference state&lt;br&gt;Compressible fluid</td>
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<td>Lorenz (1978, 1979)</td>
<td></td>
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<tr>
<td>Holliday and McIntyre (1981)</td>
<td></td>
<td>Hydrostatic reference state&lt;br&gt;Incompressible fluid</td>
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<td>Andrews (1981)</td>
<td>[ A = c_p T_0 \left[ e^\eta \left(\frac{p}{p_0}\right)^\kappa + e^\eta_\kappa \left(\frac{p}{p_0}\right)^\kappa \left(\frac{p_0}{p} - 1\right) - 1 - \eta \right] ]</td>
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<td></td>
<td>[ \eta = \frac{s - s_0}{c_p} ]</td>
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<td></td>
<td>The same as Lorenz (1955), but through a unified Hamiltonian method.</td>
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<tr>
<td>Shepherd (1993)</td>
<td>[ A = \frac{R}{2g\rho'<em>{0}} \int</em>{0}^{p_s} \frac{\theta'}{(p^{1-\kappa})(-\frac{\partial \theta}{\partial p})} \cdot \left(\frac{\partial'}{\theta}\right) dp ]</td>
<td>Hydrostatic reference state&lt;br&gt;Compressible fluid</td>
</tr>
<tr>
<td>Codoban and Shepherd (2003)</td>
<td></td>
<td>Hydrostatic axisymmetric rotating reference state&lt;br&gt;Incompressible fluid</td>
</tr>
</tbody>
</table>
| Andrews (2006)             | \[ A \equiv \tilde{C}(\mu, s) - \tilde{C}(\mu_0, s_0) - (\mu - \mu_0)\tilde{C}_\mu(\mu_0, s_0) \]  
\[ + H(s, p) - H(s_0, p_0) - \frac{p - p_0}{\rho} \] | Hydrostatic axisymmetric rotating reference state<br>Compressible fluid |

\(\tilde{A}\) means **global** APE, while \(A\) referred to **local** APE.
Future Work

- Finish the APE literature survey table. Study the relation between APE, CAPE, and static energy.

- Calculate the dry available potential energy for a real case atmosphere and a numerical simulation. Try to calculate the distribution of \((T - \hat{T})/T\) on \((r, z)\). With the aid of spectral method domain decomposition technique.

- Using the APE defined here to calculate the heating efficiency for tropical cyclone as Hack and Schubert (1986).

- Since the most general variable “Entropy” is used through our derivation, extending to the moist available potential energy with the aid of Ooyama’s thermodynamics scheme (Ooyama 1990, 2001) is viable.
Fig. 4. Tangential momentum (left set of panels) and dynamical efficiency factor $\eta(r, z)$ in the region $r = 0$ to $r = 1000$ km at 24, 48 and 72 hours for the invariantly forced balance model. The contour interval is 1 m s$^{-1}$ for the momentum field and 0.1 percent for the dynamic efficiency factor. Negative values are denoted by dashed lines. See text for further discussion.
Global frequency and distribution of total lighting flashes

\[ \text{CAPE} = R_d \int_{EL}^{LFC} (T_v' - T_v) d \ln p \]

Wallace and Hobbs (2006)
Ekman-Taylor Layer under a Circular Vortex
To derive the axisymmetric analytic solutions of Ekman layer with slip boundary condition.

Gradient wind balance
Primitive governing equations

\[
\begin{align*}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + v \frac{\partial u}{\partial \lambda} + w \frac{\partial u}{\partial z} - f v - \frac{v^2}{r} + \frac{1}{\rho} \frac{\partial p}{\partial r} &= K \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + K \frac{\partial^2 u}{\partial \lambda^2} + K \frac{\partial^2 u}{\partial z^2} \\
\frac{\partial v}{\partial t} + \frac{v}{r} \frac{\partial v}{\partial \lambda} + w \frac{\partial v}{\partial z} + \left( f + \frac{\partial (rv)}{\partial r} \right) u + \frac{1}{r} \frac{\partial p}{\partial \lambda} &= K \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v}{\partial r} \right) + K \frac{\partial^2 v}{\partial \lambda^2} + K \frac{\partial^2 v}{\partial z^2}
\end{align*}
\]

- Assume axisymmetric \( \frac{\partial}{r \partial \lambda} = 0 \)
- Assume \( u, v \) are the function of \( z \) only. \((Stationary)\)
- Assume no vertical advection in the momentum equations.
Transient inertial oscillations have been eliminated.

Using gradient wind balance:

\[ (f + \frac{v_{gr}}{r}) v_{gr} = \frac{1}{\rho} \frac{\partial p}{\partial r} \]

Assumptions: \( u'(z), v'(z) \)

Small perturbations

Compare with Holton (2004)

\[ K \frac{\partial^2 u}{\partial z^2} + f(v - v_g) = 0 \]

\[ K \frac{\partial^2 v}{\partial z^2} - f(u - u_g) = 0 \]
Literature Reviews

◆ Eliassen and Lystad (1977, hereafter EL77)
From numerical simulation with slip bottom boundary condition
Drop non-cyclostrophic terms
Maximum updraft always locates within the radius of maximum winds, and it moved outward as the vortex becomes stronger and inward as the drag coefficient increased.

◆ Montgomery et al. (2001)
Test the assumption to drop the non-cyclostrophic terms in EL77 with a Navier-Stokes model.
Cyclostrophic balance approximation appears to be quite accurate for tropical storm strength vortices.
Literature Reviews

- Kepert (2001)
  Asymmetric linear analytic solution with slip bottom boundary condition
  Relevant to a moving storm
  Ignore vertical advection

- Kepert and Wang (2001)
  Nonlinear model
  Vertical advection plays a significant role in Ekman layer.

\[
K \frac{\partial^2 u}{\partial z^2} - \frac{\bar{v}}{r} \frac{\partial u}{\partial \lambda} + \left( f + \frac{2\bar{v}}{r} \right) v = 0
\]

\[
K \frac{\partial^2 v}{\partial z^2} - \frac{\bar{v}}{r} \frac{\partial v}{\partial \lambda} - \left( f + \frac{\bar{v}}{r} + \frac{\partial \bar{v}}{\partial r} \right) u = 0
\]
Bottom Boundary Condition
Schematic Diagram

Top of the Ekman Layer

Ekman layer (the layer we are interested)

Surface (Prandtl) layer ~ 10 m

We squeeze the lowest ~10 m surface layer as the bottom boundary.
Boundary Conditions

Lower B. C.  

\[ K \left( \frac{\partial v}{\partial z} \right) = c_D \left| \mathbf{v} \right| \mathbf{v} \text{ at } z = 0 \]

Components form:

\[
\begin{align*}
K \frac{\partial u}{\partial z} &= c_D \left( u^2 + v^2 \right)^{1/2} u \\
K \frac{\partial v}{\partial z} &= c_D \left( u^2 + v^2 \right)^{1/2} v
\end{align*}
\text{ at } z = 0
\]

Upper B. C.

\[ u \to 0 \quad \text{and} \quad v \to v_{gr} \quad \text{as} \quad z \to \infty \]
Solutions

\[ u(r, z) = e^{-\gamma(r)z} \left\{ u(r, 0) \cos[\gamma(r)z] + \left( \frac{\zeta(r)}{\bar{\zeta}(r)} \right)^{1/2} \left[ v(r, 0) - v_{gr}(r) \right] \sin[\gamma(r)z] \right\} \]

\[ v(r, z) = v_{gr}(r) + e^{-\gamma(r)z} \left\{ \left[ v(r, 0) - v_{gr}(r) \right] \cos[\gamma(r)z] - \left( \frac{\zeta(r)}{\bar{\zeta}(r)} \right)^{1/2} u(r, 0) \sin[\gamma(r)z] \right\} \]

where,

\[ \bar{\zeta}(r) = f + \frac{2v_{gr}(r)}{r} \]

\[ \zeta(r) = f + \frac{\partial[r v_{gr}(r)]}{r \partial r} \]

\[ \gamma(r) = \left( \frac{\bar{\zeta}(r) \zeta(r)}{4K^2} \right)^{1/4} \]

Inertial Stability:

\[ S_I = \bar{\zeta}(r) \zeta(r) \]

Ekman Layer Depth:

\[ D_e = \frac{\pi}{\gamma} \]
Using iterative procedure to determine $u(r, 0)$ and $v(r, 0)$ at the bottom surface.
Ekman Pumping

Vertical integration of the continuity equation yields

\[
\frac{\partial (ru)}{r \partial r} + \frac{\partial w}{\partial z} = 0
\]

\[
w(r, \infty) = -\frac{\partial}{r \partial r} \int_0^\infty ru(r, z) \, dz
\]

\[
2\gamma \int_0^\infty e^{-\gamma z} \cos(\gamma z) \, dz = 2\gamma \int_0^\infty e^{-\gamma z} \sin(\gamma z) \, dz = 1
\]

Ekman Pumping Formula:

\[
w(r, \infty) = \frac{\partial}{r \partial r} \left( \frac{c_D[u^2(r, 0) + v^2(r, 0)]^{1/2} rv(r, 0)}{\zeta(r)} \right)
\]
Example

\[ f = 5.0 \times 10^{-5} \text{ s}^{-1} \]
\[ K = 5 \text{ m}^2\text{s}^{-1} \]
\[ c_D = 1.5 \times 10^{-3} \]

Vortex A

\[ v_{gr}(r) = v_{\text{max}} \left( \frac{2(r/r_{\text{max}})}{1 + (r/r_{\text{max}})^2} \right) \]

\[ \zeta(r) = f + \frac{2v_{\text{max}}}{r_{\text{max}}} \left[ 1 + \left( \frac{r}{r_{\text{max}}} \right)^2 \right]^{-2} \]
Results

(a)

(b)
The maximum updraft locates near the radius of maximum wind and moves outward as the vortex becomes stronger. This result matches the results shown in Eliassen and Lystad (1977).
Vortex B

Rozoff et al.
2006

\[ v_{gr}(r) = \frac{\Gamma}{2\pi r} \left( 1 - e^{-r^2/b^2} \right) \]

\[ r_{\text{max}} \approx 1.121b \]

\[ v_{gr}(r_{\text{max}}) \approx 0.638\Gamma/(2\pi b) \]

\[ \zeta(r) = \left[ \Gamma/(\pi b^2) \right] \exp\left(-r^2/b^2\right) \]
Results

(a) 

(b)
The maximum updraft does not locate near the radius of maximum wind but still moves outward as the vortex becomes stronger.
**Summary**

- **Ekman pumping** is very sensitive to slightly change of the vorticity distribution when we hold maximum tangential wind fixed.

- The **Ekman layer depth** increases from the center and out, and also with weakening the vortex. This results match with Eliassen and Lystad (1977)

- Different intensity of the vortex will not change θ much, and θ gets larger outward of the vortex.

- The solution validity shows nonlinear advection terms are all smaller than the sum of Coriolis force, centrifugal force, and pressure gradient force, and the gradient wind balance assumption is valid in the interior for the vortex.
Future Work

Fung (1979)
Hypothesizing that hurricane spiral bands are produced through dynamic instability of boundary-layer flow structures.

We propose to do a 3D simulation to generate an annular Ekman pumping area as the spiral bands (related to filamentation process).

We will also study the influence of PV mixing to the Ekman pumping distribution and strength.

Kuo et al. (2008)
探討已存在垂直上升運動的渦旋，其伴隨的 Ekman Pumping 之線性解析解，與非線性數值解的異同。並探討 Ekman Pumping 與渦旋可用位能、加熱效率間的關係。
Thank you!

A painting with filaments!

星夜  Van Gogh