教育部國家講座課程

颱風與渦旋動力學

Typhoon and Vortex Dynamics

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中華民國九十七年二月十九日

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九十六學年度國家講座教學課程
「颱風與渦旋動力學」

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「颱風與渦旋動力學」

課程綱要

1. Fundamentals（流力方程式, 科氏力, Levi-Civita 張量符號）
2. Thermodynamics and air and sea interaction（1^{st} and 2^{nd} laws, entropy, enthalpy, free energy; and Legendre transform）熱力學與海氣交互作用
3. Potential vorticity and quasi-equilibrium dynamics 位渦與準平衡動力
4. 2D and 3D turbulent flows 兩維與三維亂流
5. Vertical normal modes of a continues stratified fluid 垂直正模
6. The f-plane, β-plane, barotropic dynamics of typhoon, wave and instability f 面與β面颱風動力，波動與不穩度
7. The geostrophic adjustment and typhoon dynamics 地轉調節與颱風動力
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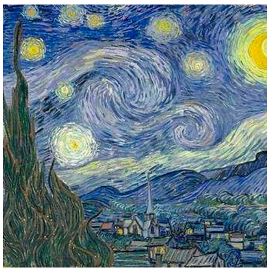
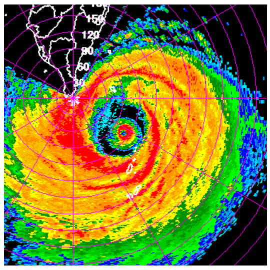
9. Boundary layer dynamics and dimensional analysis 邊界層動力與因次分析
10. Asymmetric dynamics in typhoon 颱風不對稱動力
11. Track Problem 颱風路徑問題
12. Nonlinear wave energy accumulation 非線性能量累積與颱風生成
13. Concentric eyewall formation dynamics 雙眼牆生成動力
14. Spiral bands and meso-vortex 颱風螺旋雨帶雨中小尺度渦旋
15. Vorticity stirring and mixing 渦度攪拌與渦度混合
16. Tornado and other rotating disk dynamics 龍捲風與其他旋轉體動力

教育部國家講座課程
臺大數學科學中心跨領域課程

Typhoon and Vortex Dynamics
颱風與渦旋動力學





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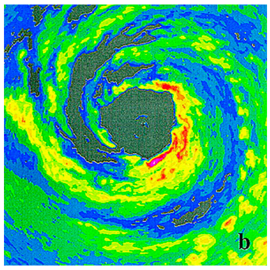
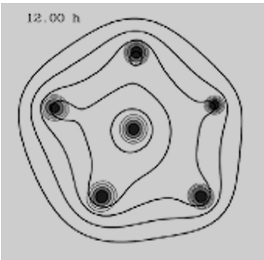
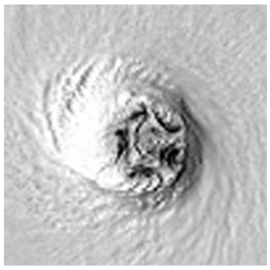
臺大大氣科學系教授



時間：**10/12-12/28 (共12週)**
星期五 **11:20-12:30**

地點：臺大新數學館308





對象：對颱風流體力學有興趣之
各大學高年級生或研究生
（不要求流體力學背景知識）

課程簡介：

颱風系統包含渦旋旋轉流體力學、水氣潛熱釋放、海氣交互作用等許多複雜的流體動力學與熱力學過程，而其現象亦涉及許多不同時間與空間尺度運動的交互作用，是相當典型的「跨尺度」數學物理問題。課程闡述颱風與渦旋動力，以「準平衡動力」瞭解颱風渦旋跨尺度的動力現象，內容包括颱風路徑動力、颱風次環流與對稱動力、非線性能量累積與颱風生成、橢圓形颱風眼動力、雙眼牆颱風渦度動力、二度空間亂流、波動與不穩度等，並旁及近十年颱風的最新研究成果與重要問題。課程將以數學公式推導與物理詮釋為主，並以數值/流力實驗動畫與簡報圖檔輔助教學。

【註】此課程無學分，課後12:30-13:00為自由討論時間，由主辦單位提供午餐，
請來信tims@tims.ntu.edu.tw或致電(02)3366-2834 臺大數科中心 易小姐 報名。

主辦單位：臺大數學科學中心
協辦單位：臺大大氣科學系、國家理論科學研究中心數學組

教育部國家講座課程—颱風與渦旋動力學
Typhoon and Vortex Dynamics

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Lecture 1

Levi-Civita Symbol and Basic Equations

(A) Levi-Civita Symbol:

$$\vec{v} = u_i \tag{1.1}$$

$$\nabla \cdot \vec{v} = \frac{\partial u_m}{\partial x_m} \tag{1.2}$$

$$\nabla = \frac{\partial}{\partial x_i} \tag{1.3}$$

$$\vec{v} \cdot \nabla = u_j \frac{\partial}{\partial x_j} \tag{1.4}$$

$$\nabla \times \vec{v} = \varepsilon_{ijk} \, \frac{\partial}{\partial x_j} v_k \tag{1.5}$$

$$\nabla^2 = \frac{\partial^2}{\partial x_i^2} \tag{1.6}$$

$$\varepsilon_{ijk} \, \varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \tag{1.7}$$

$$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C}) \tag{1.8}$$

(B) Momentum and Continuity Equation:

Mass conservation (v_m is an arbitrary material volumn):

$$\frac{d}{dt} \int_{v_m} dv = \int_{\partial v_m} \vec{v} \cdot d\vec{s} = \int_{v_m} \nabla \cdot \vec{v} \; dv \tag{1.9}$$

$$\lim_{\Delta\to 0}\frac{1}{\Delta v}\frac{d\Delta v}{dt}=\nabla\cdot\vec{v} \tag{1.10}$$

$$\frac{d}{dt}\int_{v_m}\rho\,dv=\int_{v_m}\frac{d\rho}{dt}\,dv+\int_{v_m}\rho\nabla\cdot\vec{v}\,dv=0 \tag{1.11}$$

Continuity equation (advective form):

$$\frac{d\rho}{dt}+\rho\nabla\cdot\vec{v}=0 \tag{1.12}$$

Continuity equation (flux form):

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\vec{v})=0 \tag{1.13}$$

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\frac{dx_i}{dt}\frac{\partial}{\partial x_i}=\frac{\partial}{\partial t}+u_j\frac{\partial}{\partial x_j}=\frac{\partial}{\partial t}+\vec{v}\cdot\nabla \tag{1.14}$$

Momentum equation:

$$\frac{d}{dt}\int_{v_m}\rho\vec{v}\,dv=-\int_{\partial v_m}p\,d\vec{s} \tag{1.15}$$

$$\int_{v_m}\rho\frac{d\vec{v}}{dt}\,dv=-\int_{v_m}\nabla p\,dv \tag{1.16}$$

$$\rho\,\frac{d\vec{v}}{dt}=-\nabla p \tag{1.17}$$

Diffusion:

$$\nabla\cdot\nu\,(\nabla\vec{v})=\nu\,\nabla^2\vec{v}, \tag{1.18}$$

where $\nu\;(\nabla\vec{v})$ is the momentum flux by the diffusion process and is proportional to the momentum gradient.

$$\frac{\partial\vec{v}}{\partial t}+\vec{v}\cdot\nabla\vec{v}=-\frac{1}{\rho}\nabla p-\nabla\Phi \tag{1.19}$$

$$\vec{\zeta}\times\vec{u}=\varepsilon_{nim}\,\varepsilon_{ijk}\,\frac{\partial u_k}{\partial x_j}\,u_m=u_j\,\frac{\partial u_i}{\partial x_j}+\nabla\frac{u_i^2}{2}=\vec{v}\cdot\nabla\vec{v}+\nabla K \tag{1.20}$$

Lagrange (1781) showed:

$$\frac{\partial\vec{u}}{\partial t}+\vec{\zeta}\times\vec{u}=-\frac{1}{\rho}\nabla p-\nabla K-\nabla\Phi \tag{1.21}$$

with

$$K = \frac{{u_i}^2}{2} \tag{1.22}$$

$$\Phi = gz \tag{1.23}$$

∇K and $\nabla \Phi$ are irrotational because $\nabla \times \nabla() = 0$.

(C) Constitutive Equation:

Barotropic fluid (irrotational):

$$p = p\,(\rho) \tag{1.24}$$

Baroclinic fluid (rotational):

$$p = p\,(\rho, \theta) \tag{1.25}$$

Ideal gas law:

$$p = \rho RT \tag{1.26}$$

$$p\alpha = RT \tag{1.27}$$

where $R = 287\ J/(K \cdot kg)$ is the dry air gas constant.

(D) Thermodynamic Equations:

◊ First law:

$$\delta Q = d\,U + \delta W \tag{1.28}$$

$$Tds = c_p\,dT + p\,d\alpha \tag{1.29}$$

where $c_p = 1004\ J/(K \cdot kg)$ is the specific heat at constant pressure.

$$Tds = c_p\,dT - \alpha\,dp \tag{1.30}$$

$$c_p\,dT = \alpha\,dp \tag{1.31}$$

$$\int_{\theta}^{T} \frac{dT'}{T'} = \int_{p_0}^{p} \frac{R}{c_p} \frac{dp'}{p'} \tag{1.32}$$

Potential temperature:

$$\theta = T\ (\frac{p_0}{p})^{\kappa} \tag{1.33}$$

$$\kappa = \frac{R}{c_p} \tag{1.34}$$

◊ Application of the second law:

$$Q_1 = W + Q_2 \tag{1.35}$$

$$\frac{Q_2}{T_2} > \frac{Q_1}{T_1} \tag{1.36}$$

$$\frac{Q_1 - W}{T_2} > \frac{Q_1}{T_1}$$

$$Q_1 \left[\frac{1}{T_2} - \frac{1}{T_1}\right] > \frac{W}{T_2}$$

$$\frac{W}{Q_1} < \frac{T_1 - T_2}{T_1} \tag{1.37}$$

Assume $T_1 \sim 300K$, $\Delta T \sim 100K$, the efficiency of the atmosphere is only 1/3.

(E) Vertical Coordinate Transformation:

$$f(X_i, s, T) = f(x_i, z, t) \tag{1.38}$$

$$\frac{\partial f}{\partial X_i} = \frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial X_i} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial X_i} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial X_i} = \frac{\partial f}{\partial x_i} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial X_i} \tag{1.39}$$

$$\frac{\partial p}{\partial X_i} = \frac{\partial p}{\partial x_i} + \frac{\partial p}{\partial z}\frac{\partial z}{\partial X_i} \tag{1.40}$$

if $s = p$, then $\frac{\partial p}{\partial X_i} = 0$,

$$\frac{1}{\rho}\frac{\partial p}{\partial x_i} = \frac{\partial \Phi}{\partial X_i} \tag{1.41}$$

if $s = \theta$,

$$\frac{1}{\rho}\frac{\partial p}{\partial x_i} = \frac{\partial M}{\partial X_i} \tag{1.42}$$

where $M \equiv c_p T + \Phi$ is *Montgomery Streamfunction.*

Lecture 2

Vorticity and Potential Vorticity Equation

(A) Vorticity Equation Derivation.

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\frac{1}{\rho} \nabla p - \nabla \Phi \tag{2.1}$$

$$\vec{v} \cdot \nabla \vec{v} = \vec{\zeta} \times \vec{v} + \nabla K \tag{2.2}$$

$$\frac{\partial \vec{v}}{\partial t} + \vec{\zeta} \times \vec{v} = -\frac{1}{\rho} \nabla p - \nabla (\Phi + K) \tag{2.3}$$

$$\vec{\zeta} = \nabla \times \vec{v} \tag{2.4}$$

$$\nabla \times (\vec{\zeta} \times \vec{v}) = \vec{v} \cdot \nabla \vec{\zeta} + \vec{\zeta}\, \nabla \cdot \vec{v} - \vec{\zeta} \cdot \nabla \vec{v} \tag{2.5}$$

$$\frac{\partial \vec{\zeta}}{\partial t} + \vec{v} \cdot \nabla \vec{\zeta} + \vec{\zeta}\, \nabla \cdot \vec{v} = \vec{\zeta} \cdot \nabla \vec{v} + \vec{B} \tag{2.6}$$

$$\vec{B} = \nabla \times (-\frac{1}{\rho} \nabla p) \tag{2.7}$$

Vorticity equation derivation with Levi-Civita symbol.

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} - \frac{\partial \Phi}{\partial x_i} \tag{2.8}$$

$$\frac{\partial u_i}{\partial t} + \varepsilon_{ijk}\, \zeta_j\, v_k = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} - \frac{\partial \Phi + K}{\partial x_i} \tag{2.9}$$

$$\frac{\partial \zeta_i}{\partial t} + u_j \frac{\partial \zeta_i}{\partial x_j} + \zeta_i\, \frac{\partial u_j}{\partial x_j} = \zeta_j\, \frac{\partial u_i}{\partial x_j} + B_i \tag{2.10}$$

(B) Potential Vorticity Equation Derivation.

(1) Derivation with Levi-Civita Symbol

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho\, u_j}{\partial x_j} = 0 \tag{2.11}$$

$$\frac{\partial \psi}{\partial t} + u_j \frac{\partial \psi}{\partial x_j} = \dot{\psi} \tag{2.12}$$

According to (2.10) with non-divergence:

$$\frac{\partial \zeta_i/\rho}{\partial t} + u_j \frac{\partial \zeta_i/\rho}{\partial x_j} = \frac{\zeta_j}{\rho}\,\frac{\partial u_i}{\partial x_j} + \frac{B_i}{\rho} \tag{2.13}$$

According to (2.12):

$$\frac{\partial}{\partial t}\frac{\partial \psi}{\partial x_i} + u_j \frac{\partial}{\partial x_j}\frac{\partial \psi}{\partial x_i} + \frac{\partial u_j}{\partial x_i}\frac{\partial \psi}{\partial x_j} = \frac{\dot{\psi}}{\partial x_i} \tag{2.14}$$

$(2.13) \cdot \frac{\partial \psi}{\partial x_i} + (2.14) \cdot \frac{\zeta_i}{\rho}$:

$$\frac{\partial}{\partial t}(\frac{\zeta_i}{\rho}\frac{\partial \psi}{\partial x_i}) + u_j \frac{\partial}{\partial x_j}(\frac{\zeta_i}{\rho}\frac{\partial \psi}{\partial x_i}) = \frac{\zeta_i}{\rho}\frac{\partial \dot{\psi}}{\partial x_i} + \frac{B_i}{\rho}\frac{\partial \psi}{\partial x_i} \tag{2.15}$$

(2) Heuristic Approach

$$\frac{d}{dt}\oint \vec{v} \cdot d\vec{l} = \frac{d}{dt}\iint \vec{\zeta} \cdot \widehat{n}\, dA \tag{2.16}$$

$$\frac{d}{dt}\oint \vec{v} \cdot d\vec{l} = \oint \frac{d\vec{v}}{dt} \cdot d\vec{l} = \oint -\frac{1}{\rho}\nabla p \cdot d\vec{l} = \iint \vec{B} \cdot \widehat{n}\, dA \tag{2.17}$$

$$\frac{d}{dt}(\vec{\zeta} \cdot \widehat{n}\ \Delta A) = \vec{B} \cdot \widehat{n}\ \Delta A \tag{2.18}$$

$$\frac{1}{\vec{\zeta} \cdot \widehat{n}}\frac{d\,\vec{\zeta} \cdot \widehat{n}}{dt} + \frac{1}{\Delta A}\frac{d\,\Delta A}{dt} = \frac{\vec{B} \cdot \widehat{n}}{\vec{\zeta} \cdot \widehat{n}} \tag{2.19}$$

$$\frac{d}{dt}(\frac{\vec{\zeta} \cdot \widehat{n}}{\rho \Delta h}) = \frac{\vec{B} \cdot \widehat{n}}{\rho \Delta h} \tag{2.20}$$

$$\nabla \psi = \frac{\Delta \psi}{\Delta h}\,\widehat{n} \tag{2.21}$$

$$\frac{d}{dt}(\frac{\vec{\zeta} \cdot \nabla \psi}{\rho \Delta \psi}) = \frac{\vec{B} \cdot \nabla \psi}{\rho \Delta \psi} \tag{2.22}$$

Define

$$P = \frac{\vec{\zeta} \cdot \nabla \psi}{\rho} \tag{2.23}$$

$$\frac{d}{dt}(\frac{P}{\Delta \psi}) = \frac{\vec{B} \cdot \nabla \psi}{\rho \Delta \psi} \tag{2.24}$$

Expand the left hand side of (2.24), and use the relationship of (2.21), we could obtain the complete potential vorticity equation.

$$\frac{dP}{dt} = \frac{\vec{\zeta}}{\rho} \cdot \nabla \dot{\psi} + \frac{\vec{B} \cdot \nabla \psi}{\rho} \tag{2.25}$$

(C) Hydrostatic Approximation

Quasi-static $\longrightarrow$ hydrostatic adjustment

$$\frac{d\vec{v}}{dt} + f\widehat{k} \times \vec{v} = -\frac{1}{\rho_0} \nabla p \tag{2.26}$$

$$UN \quad fU \quad \frac{p'}{\rho_0 L}$$

$$\rho \frac{dw}{dt} = \frac{\partial p}{\partial z} - \rho g \tag{2.27}$$

$$\rho_0 WN \quad \frac{p'}{H}$$

(1)

$$UN \sim \frac{p'}{\rho_0 L}$$

$$p' \sim \rho_0 LUN$$

$$\rho_0 WN \ll \frac{p'}{H} \sim \frac{\rho_0 LUN}{H}$$

$$\frac{WH}{LU} \ll 1$$

$$\frac{\frac{W}{H}}{\frac{U}{L}}(\frac{H^2}{L^2}) \ll 1 \tag{2.28}$$

(2)

$$p' \sim \rho_0 L f U$$

$$\rho_0 W N \ll \frac{\rho_0 L f U}{H}$$

$$\frac{\frac{W}{H}}{\frac{U}{L}}(\frac{H^2}{L^2})\frac{N}{f} \ll 1 \tag{2.29}$$

(D) PV in Hydrostatic Atmosphere with θ Coordinate

$$PV = \frac{\vec{\zeta} \cdot \nabla\theta}{\rho} \tag{2.30}$$

$$\rho \cdot PV = -\frac{\partial v}{\partial z}\frac{\partial \theta}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial \theta}{\partial y} + (f + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})\frac{\partial \theta}{\partial z} \tag{2.31}$$

$$\frac{\partial v}{\partial x} = \left(\frac{\partial v}{\partial x}\right)_\theta + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x} \tag{2.32}$$

$$\frac{\partial u}{\partial y} = \left(\frac{\partial u}{\partial y}\right)_\theta + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} \tag{2.33}$$

$$\frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial z} = \frac{\partial v}{\partial z} \tag{2.34}$$

$$\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial z} = \frac{\partial u}{\partial z} \tag{2.35}$$

Substitute (2.32) $\sim$ (2.35) into (2.31), we could obtain:

$$\begin{aligned} PV &= \left[f + \left(\frac{\partial v}{\partial x}\right)_\theta - \left(\frac{\partial u}{\partial y}\right)_\theta\right]\frac{\partial \theta}{\partial z}\frac{1}{\rho} \\ &= \frac{f + \left(\frac{\partial v}{\partial x}\right)_\theta - \left(\frac{\partial u}{\partial y}\right)_\theta}{-\frac{1}{g}\frac{\partial p}{\partial \theta}} \end{aligned} \tag{2.36}$$

Analog to shallow water PV.

Ertel's Derivation Potential Vorticity

$$\frac{\partial \zeta_i}{\partial t} + u_j \frac{\partial \zeta_i}{\partial x_j} + \zeta_i \frac{\partial u_j}{\partial x_j} = \zeta_j \frac{\partial u_i}{\partial x_j} + B_i$$ **Helmholtz Vorticity Equation (1858)**

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0$$ **Euler Continuity Equation (1750)**

$$\frac{d}{dt}\left(\frac{\zeta_i}{\rho}\right) = \frac{\zeta_j}{\rho}\frac{\partial u_i}{\partial x_j} + \frac{B_i}{\rho}$$

$$\frac{d}{dt}\psi = \dot{\psi}$$ **(some scalar function ψ)**

$$\frac{d}{dt}\frac{\partial \psi}{\partial x_i} = -\frac{\partial u_j}{\partial x_i}\frac{\partial \psi}{\partial x_j} + \frac{\partial \dot{\psi}}{\partial x_i}$$

$$\frac{\zeta_i}{\rho}\frac{d}{dt}\frac{\partial \psi}{\partial x_i} = -\frac{\zeta_i}{\rho}\frac{\partial u_j}{\partial x_i}\frac{\partial \psi}{\partial x_j} + \frac{\zeta_i}{\rho}\frac{\partial \dot{\psi}}{\partial x_i}$$

$$+)\ \frac{\partial \psi}{\partial x_i}\frac{d}{dt}\left(\frac{\zeta_i}{\rho}\right) = \frac{\zeta_j}{\rho}\frac{\partial u_i}{\partial x_j}\frac{\partial \psi}{\partial x_i} + \frac{B_i}{\rho}\frac{\partial \psi}{\partial x_i} \qquad \left(\frac{B_i}{\rho}\frac{\partial \psi}{\partial x_i} = \frac{1}{\rho}N(\rho, P, \psi)\right)$$

$$\frac{d}{dt}\left(\frac{\zeta_i}{\rho}\frac{\partial \psi}{\partial x_i}\right) = \frac{\zeta_i}{\rho}\frac{\partial \dot{\psi}}{\partial x_i} + \frac{1}{\rho}N(\rho, P, \psi)$$

Potential Vorticity

$$\frac{d}{dt}\int \vec{V}\cdot d\vec{l} = -\oint \frac{dp}{\rho}$$ **Kelvin theorem**

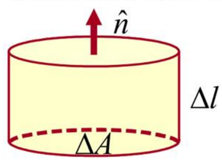
$$\frac{d}{dt}(\nabla\times\vec{V}\cdot\hat{n}\,\Delta A) = \vec{B}\cdot\hat{n}\,\Delta A \qquad \vec{B} = \nabla\times\left(-\frac{1}{\rho}\nabla p\right)$$ **Stokes' theorem**

$$\frac{d}{dt}\left(\frac{\nabla\times\vec{V}\cdot\hat{n}}{\rho\,\Delta l}\right) = \frac{\vec{B}\cdot\hat{n}}{\rho\,\Delta l}$$

$$\frac{d}{dt}\left(\frac{\nabla\times\vec{V}\cdot\nabla\psi}{\rho}\right) = \frac{\nabla\times\vec{V}\cdot\nabla\psi}{\rho\,\Delta\psi}\frac{d\Delta\psi}{dt} + \frac{\vec{B}\cdot\nabla\psi}{\rho}$$

$$\frac{d}{dt}\left(\frac{\nabla\times\vec{V}\cdot\nabla\psi}{\rho}\right) = \frac{\nabla\times\vec{V}}{\rho}\cdot\nabla\frac{d\psi}{dt} + \frac{\vec{B}\cdot\nabla\psi}{\rho}$$

$$\frac{d}{dt}(\rho\,\Delta A\,\Delta l) = 0$$ **Euler's mass conservation**



$$\nabla\psi = \frac{\Delta\psi}{\Delta l}\hat{n}$$

$$\frac{\nabla\times\vec{V}\cdot\hat{n}}{\rho\,\Delta l} = \frac{\nabla\times\vec{V}\cdot\nabla\psi}{\rho\,\Delta\psi}$$

The conservation of PV is really the Kelvin theorem with a special but useful contour.

Lecture 3

Vertical Transform and Shallow Water Equation

(A) Vertical transform:

$$s = s(p) \equiv \frac{c_p \theta_0}{g} \left(1 - \left(\frac{p}{p_0}\right)^\kappa\right) \tag{3.1}$$

$$\kappa = \frac{R}{c_p} \tag{3.2}$$

$$b = \frac{g}{\theta_0} \theta \tag{3.3}$$

Linearized equation in s coordinate (with J represents diabatic heating).

$$\frac{\partial \vec{v}}{\partial t} + f \, \widehat{k} \times \vec{v} = -\nabla \Phi \tag{3.4}$$

$$\nabla \cdot \vec{v} + \frac{\partial \dot{s}}{\partial s} = 0 \tag{3.5}$$

$$\frac{\partial \Phi}{\partial s} = b \tag{3.6}$$

$$\frac{\partial b}{\partial t} + \dot{s} \, N^2 = J \tag{3.7}$$

Boundary condition: $w = dz/dt = 0$ at $s = 0$, $\dot{s} = d\,p/dt = 0$ at $s = H$.
$\frac{\partial \tilde{\Phi}}{\partial t}$ is the local height change that can be resulted from diabatic heating.

Boundary condition derivation.

$$J = \frac{\partial}{\partial s}\left(\frac{\partial \tilde{\Phi}}{\partial t}\right) \tag{3.8}$$

Substitute (3.6) and (3.8) into (3.7) we could obtain:

$$\dot{s} = -\frac{1}{N^2}\frac{\partial}{\partial s}(\frac{\partial \Phi}{\partial t} - \frac{\partial \tilde{\Phi}}{\partial t}) \tag{3.9}$$

At $s = 0$, $w = dz/dt = 0$,

$$\frac{d\Phi}{dt} = 0 \tag{3.10}$$

$$\frac{\partial \Phi}{\partial t} + \dot{s}\frac{\partial \bar{\Phi}}{\partial s} = 0 \tag{3.11}$$

$$\frac{\partial \Phi}{\partial t} + b_0 \; \dot{s} = 0 \tag{3.12}$$

Substitute (3.9) into (3.12), and with $\frac{\partial \tilde{\Phi}}{\partial t} = 0$ at $s = 0$ we could obtain:

$$(\frac{\partial \Phi}{\partial t} - \frac{\partial \tilde{\Phi}}{\partial t}) - \alpha\frac{\partial}{\partial s}(\frac{\partial \Phi}{\partial t} - \frac{\partial \tilde{\Phi}}{\partial t}) = 0 \tag{3.13}$$

where

$$\alpha = \frac{b_0}{N^2} \tag{3.14}$$

At $s = H$, $\dot{s} = d\,p/dt = 0$,

$$\frac{\partial}{\partial s}(\frac{\partial \Phi}{\partial t} - \frac{\partial \tilde{\Phi}}{\partial t}) = 0 \tag{3.15}$$

Inner product:

$$\frac{1}{H}\int_0^H uv \; ds =< u, v > \tag{3.16}$$

$$< \mathcal{L}u, v >=< u, \mathcal{L}v > \tag{3.17}$$

Based on the Sturm-Liouville theorem, the basis function is complete, orthogonal, and the eigenvalue is real.

$$\mathcal{L}\Psi_n = \lambda_n \Psi_n \tag{3.18}$$

$$\Psi_n - \alpha\frac{\partial \Psi_n}{\partial s} = 0 \tag{3.19}$$

$$\frac{\partial \Psi_n}{\partial s} = 0 \tag{3.20}$$

If $\lambda_n = \frac{1}{c_n^2} > 0$, we could obtain the shallow water equation.

$$\frac{\partial \vec{v_n}}{\partial t} + f \; \widehat{k} \times \vec{v_n} = -\nabla \Phi_n \tag{3.21}$$

$$\frac{\partial \Phi}{\partial t} + {c_n}^2 \nabla \cdot \vec{v_n} = \frac{\partial \tilde{\Phi}}{\partial t} \tag{3.22}$$

(B) Shallow Water Equation

$$\frac{\partial u}{\partial t} - (f + \zeta)v = -\frac{\partial \Phi}{\partial x} - \frac{\partial \frac{u^2+v^2}{2}}{\partial x} \tag{3.23}$$

$$\frac{\partial v}{\partial t} + (f + \zeta)u = -\frac{\partial \Phi}{\partial y} - \frac{\partial \frac{u^2+v^2}{2}}{\partial y} \tag{3.24}$$

$$\frac{\partial \Phi}{\partial t} + \frac{\partial u\Phi}{\partial x} + \frac{\partial v\Phi}{\partial y} = source/sink \tag{3.25}$$

Linearized shallow water model:

$$\frac{\partial \mathbf{u}}{\partial t} + A\mathbf{u} = \mathbf{f} \tag{3.26}$$

$$AE = ED \tag{3.27}$$

where E is the eigenvector matrix, and D is the diagonal eigenvalue matrix.

$$E^H AE = D \tag{3.28}$$

where E^H is Hermitian of E.

$$E^H \frac{\partial \mathbf{u}}{\partial t} + E^H AEE^H \mathbf{u} = E^H \mathbf{f} \tag{3.29}$$

$$\frac{\partial \mathbf{v}}{\partial t} + D\mathbf{v} = \mathbf{h} \tag{3.30}$$

(C) A Very General Dynamic System:

$$\frac{dq}{dt} + i\nu q + \alpha q = f_0 e^{i\Omega t} \tag{3.31}$$

Particular solution:

$$q_p = ae^{i\Omega t} \tag{3.32}$$

Substitute (3.32) into (3.31) we could obtain

$$a = \frac{f_0}{i\Omega + i\nu + \alpha} \tag{3.33}$$

The whole solution is

$$q(t) = \underbrace{\big(q(0) - \frac{f_0}{i\Omega + i\nu + \alpha}\big)e^{-i\nu t - \alpha t}}_{memory} + \underbrace{\frac{f_0 \; e^{i\Omega t}}{i\Omega + i\nu + \alpha}}_{quasi-equilibrium} \tag{3.34}$$

Define the response function:

$$R = \frac{1}{i\Omega + i\nu + \alpha} \tag{3.35}$$

(3.34) reduces to

$$q(t) = \big(q(0) - Rf_0\big)e^{-i\nu t - \alpha t} + f_0 \; e^{i\Omega t} R \tag{3.36}$$

(D) Normal Mode Analysis
Linear Shallow Water Equation:

$$\frac{\partial u}{\partial t} - v = -\frac{\partial \phi}{\partial x}$$

$$\frac{\partial v}{\partial t} + u = -\frac{\partial \phi}{\partial y}$$

$$\frac{\partial \phi}{\partial t} + (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) = 0 \tag{3.37}$$

Transform to spectral space,

$$\frac{d \; \widehat{u}_{kl}}{dt} - \widehat{v}_{kl} + ik\widehat{\phi}_{kl} = 0$$

$$\frac{d \; \widehat{v}_{kl}}{dt} + \widehat{u}_{kl} + il\widehat{\phi}_{kl} = 0$$

$$\frac{d \; \widehat{\phi}_{kl}}{dt} + ik\widehat{u}_{kl} + il\widehat{v}_{kl} = 0 \tag{3.38}$$

In the matrix form, (3.38) can be witten into more compact form:

$$\frac{d}{dt}\mathbf{u} + A\mathbf{u} = 0 \tag{3.39}$$

where

$$\mathbf{u} = \begin{pmatrix} \widehat{u}_{kl} \\ \widehat{v}_{kl} \\ \widehat{\phi}_{kl} \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & -1 & ik \\ 1 & 0 & il \\ ik & il & 0 \end{pmatrix}$$

$$A\psi^{(r)} = -i\nu_{(r)}\psi^{(r)} \tag{3.40}$$

1) $\nu_{(0)}$=0

$$\psi^{(0)} = \frac{1}{\nu}(il, -ik, -1)^T \tag{3.41}$$

2) $\nu_{(1)}$=ν

$$\psi^{(1)} = \frac{1}{\mu\nu\sqrt{2}}(\nu k + il, \nu l - ik, \mu^2)^T \tag{3.42}$$

3) $\nu_{(2)}$=-ν

$$\psi^{(2)} = \frac{1}{\mu\nu\sqrt{2}}(-\nu k + il, -\nu l - ik, \mu^2)^T \tag{3.43}$$

where $\mu^2 = k^2 + l^2$ and $\nu^2 = 1 + k^2 + l^2$.

$$\mathbf{u} = \alpha_1\psi^{(0)} + \alpha_2\psi^{(1)}e^{i\nu t} + \alpha_3\psi^{(2)}e^{-i\nu t} \tag{3.44}$$

The first term at the right hand side represents the stationary solution in the end, and the second and third terms represent the wave process.

$$\psi^{(0)^H}\mathbf{u} = ik\widehat{v}_{kl} - il\widehat{u}_{kl} - \widehat{\phi}_{kl} \tag{3.45}$$

The right hand side of (3.45) is the Fourier transform of Potential Vorticity. So, The gravity-inertia waves have zero potential vorticity and gravity-inertia waves are invisible on potential vorticity maps.

(E) Shallow Water Equation with Forcing

$$\frac{\partial u}{\partial t} - v = -\frac{\partial h}{\partial x} \tag{3.46}$$

$$\frac{\partial v}{\partial t} + u = 0 \tag{3.47}$$

$$\frac{\partial h}{\partial t} + \frac{\partial u}{\partial x} = Q\alpha^2 t\, e^{-\alpha t} \tag{3.48}$$

where $Q = Q(x)$, $u(0,x) = \frac{\partial u(0,x)}{\partial t} = 0$, $v(0,x) = 0$, and $h(0,x) = 0$.

$$\int_0^\infty \alpha^2\, t\, e^{-\alpha t}\, dt = 1 \tag{3.49}$$

$$\int_0^\infty \alpha^{n+1}\, t^n\, e^{-\alpha t}\, dt = 1 \tag{3.50}$$

$\frac{\partial}{\partial t}(3.46) + (3.47) - ik(3.48)$:

$$\frac{\partial^2 u}{\partial t^2} + u - \frac{\partial^2 u}{\partial x^2} = -\frac{dQ}{dx}\alpha^2 t e^{-\alpha t} \tag{3.51}$$

$$\frac{d^2 \widehat{u}}{dt^2} + (1+k^2)\widehat{u} = -ik\widehat{Q}\alpha^2 t e^{-\alpha t} \tag{3.52}$$

$$\widehat{u}(k,t) = \frac{-ik\widehat{Q}(k)}{\alpha^2+\nu^2}\alpha^2 t e^{-\alpha t} - \frac{2ik\alpha^3\widehat{Q}(k)}{(\alpha^2+\nu^2)^2}e^{-\alpha t}$$

$$+\frac{k\widehat{Q}(k)}{2\nu}\left[\frac{\alpha^2(\alpha+i\nu)^2}{(\alpha^2+\nu^2)^2}e^{-i\nu t} - \frac{\alpha^2(\alpha-i\nu)^2}{(\alpha^2+\nu^2)^2}e^{i\nu t}\right] \tag{3.53}$$

where $\nu = 1 + k^2$.
Slow forcing, $\alpha << 1$,

$$\widehat{u}(k,t) \approx \frac{-ik\widehat{Q}(k)}{1+k^2}\alpha^2 t e^{-\alpha t} \tag{3.54}$$

is the equilibrium solution.

Balanced Vortex Model and the Transverse Circulation Equation

Hung-Chi Kuo, 12/11/2007

Consider the cylindrical coordinate (r, ϕ, z, t) with

$$z = H \ln\left(\frac{p}{p_0}\right),$$

where

$$H = \frac{RT_0}{g},$$

is the constant scale height, and $p_0 = 100kPa$, $T_0 = 300K$.

The inviscid axisymmetric (i.e. $\partial\phi = 0$) balanced vortex model equations on an f plane contain (1) the gradient wind balance equation

$$(f + \frac{v}{r})v = \frac{\partial\Phi}{\partial r},$$

(2) the angular momentum conservation equation

$$\frac{Dv}{Dt} + (f + \frac{v}{r})u = 0,$$

(3) the hydrostatic approximation equation

$$\frac{\partial\Phi}{\partial z} = \frac{g}{T_0}T,$$

(4) the continuity equation

$$\frac{\partial(ru)}{r\partial r} + \frac{\partial w}{\partial z} - \frac{w}{H} = 0,$$

and (5) the thermodynamic equation

$$\frac{DT}{Dt} + \frac{RT}{c_p H}w = \frac{Q}{c_p},$$

where u and v are the radial and azimuthal component of velocity, $w = Dz/Dt$ the "log pressure vertical velocity," Φ the geopotential, T the temperature, f the constant Coriolis parameter, Q the diabatic heating (radiation and the latent heat release), and $D/Dt = \partial/\partial t + u(\partial/\partial r) + w(\partial/\partial z)$ the material derivative. Consider the continuity equation, the transverse circulation (u, w) can be expressed in terms of the streamfunction ψ such that

$$e^{-z/H}u = -\frac{\partial\psi}{\partial z}$$

$$e^{-z/H}w = -\frac{\partial(r\psi)}{r\partial r}$$

Thermal wind balance is

$$(f + \frac{2v}{r})(\frac{\partial v}{\partial z}) = \frac{g}{T_0}\frac{\partial T}{\partial r}.$$

Eliminating the local time derivatives in the tangential wind equation ($\partial v/\partial t$) and the thermodynamic equation through the use of the thermal wind balance constraint, the resulting transverse circulation is

$$\frac{\partial}{\partial r}\left(A\frac{(r\psi)}{r\partial r} + B\frac{\partial \psi}{\partial z}\right) + \frac{\partial}{\partial z}\left(B\frac{(r\psi)}{r\partial r} + C\frac{\partial \psi}{\partial z}\right) = \frac{g}{c_p T_0}\frac{\partial Q}{\partial r},$$

where the static stability A, the baroclinity B, and the inertial stability C are defined by

$$A = e^{z/H}\frac{g}{T_0}\left(\frac{\partial T}{\partial z} + \frac{RT}{c_p H}\right),$$

$$B = -e^{z/H}\frac{g}{T_0}\frac{\partial T}{\partial r} = -e^{z/H}\left(f + \frac{2v}{r}\right)\frac{\partial v}{\partial z},$$

$$C = e^{z/H}\left(f + \frac{2v}{r}\right)\left(f + \frac{\partial(rv)}{r\partial r}\right).$$

The transverse equation is elliptical if $AC - B^2 > 0$ (i.e., potential vorticity positive) everywhere. The transverse circulation equation diagnoses the amount of u and w (unbalanced flow) required to maintain the thermal wind balance when the balanced dynamics evolve with time.

The boundary conditions are: $\psi = 0$ at $r = 0$ and $r\psi \to 0$ as $r \to \infty$. The effect of boundary layer friction and Ekman pumping could be incorporated through specification of ψ at the lower boundary of the free atmosphere (i.e., at the top of the boundary layer). If our focus is on the transverse circulation induced by the diabatic heating Q, we may require that ψ vanish at both the bottom and top isobaric surfaces $z = 0, z_T$.

Both the prognostic equations of v and T are used in the derivation of the transverse equation. Thus, only one of the original prognostic equation should be used with the transverse circulation equation. Choosing to predict the mass field rather than the rotational wind field, we can write the thermodynamic equation as

$$\frac{\partial T}{\partial t} = \frac{Q}{c_p} - \frac{T_0}{g}\left(A\frac{\partial(r\psi)}{r\partial r} + B\frac{\partial \psi}{\partial z}\right).$$

After we solve the transverse circulation equation for the streamfunction ψ, we could use this thermodynamic equation to obtain the implied temperature tendency. The tangential wind v can be diagnosed through the thermal wind balance equation.

Note 1: The continuity equation is in the so called "anelastic" form. The anelastic continuity equation allows no acoustic waves in the system. The anelastic continuity equation can be derived from the continuity equation by setting $\rho = \rho_0(z)$. The vanish of $\partial\rho/\partial t$ can be achieved in the formal analysis by setting the time scale to the buoyancy time scale N^{-1}, where $N^2 = (g/\rho)(d\rho/dz)$ is the Brunt Vaisala frequency. With $\partial\rho/\partial t = 0$, we now have the anelastic continuity equation

$$\frac{\partial \rho_0 r u}{r\partial r} + \frac{\partial \rho_0 v}{r\partial \phi} + \frac{\partial \rho_0 w}{\partial z} = 0,$$

or

$$\frac{\partial r u}{r\partial r} + \frac{\partial v}{r\partial \phi} + \frac{1}{\rho_0}\frac{\partial \rho_0 w}{\partial z} = 0.$$

The equation can also be written as

$$\frac{\partial r u}{r\partial r} + \frac{\partial v}{r\partial \phi} + \frac{\partial w}{\partial z} + \frac{1}{\rho_0}\frac{d\rho_0}{dz}w = 0.$$

The last term is $-w/H$ when we have $\rho_0 = \rho(0)e^{-z/H}$. When shallow convection (i.e., the vertical scale is much smaller than the scale height H) are considered, the last term of above equation can be neglected. We then have the "incompressible" or "Boussinesq approximation",

$$\frac{\partial r u}{r\partial r} + \frac{\partial v}{r\partial \phi} + \frac{\partial w}{\partial z} = 0.$$

Note that the continuity equation in general is $D\rho/Dt + \rho\nabla\cdot\mathbf{V} = 0$ and the Boussinesq approximation is $\nabla\cdot\mathbf{V} = 0$. Thus, we can treat the $D\rho/Dt = 0$ equation in the incompressible or Boussinesq fluid as the thermodynamic equation. An incompressible fluid is just incompressible (unsqueezable and no material volume change), the density within the fluid can vary greatly and the buoyancy effect $-g\rho'/\rho$ can be very important. The Boussinesq or anelastic fluid allow no acoustic waves and thus simplified the multiple scale problem greatly. The incompressible fluid in Cartesian coordinate is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

An important question in the atmospheric and oceanic fluid dynamics is that in case of a significant $\partial u/\partial x$, what will be the dominant balance of the equation? In cases of strong rotation and strong internal stratification, the dominant balance of $\partial v/\partial y$ to the $\partial u/\partial x$ resulted in the horizontal quasi-nondivergent flow and the two-dimensional turbulence. The two-dimensional turbulence, which may be stemmed either from the fluid rotation or from the fluid internal stratification, has been a paradigm for geophysical fluid dynamics for many years. It is a striking fact that for any type of random initial state or external forcing, a two-dimensional fluid will rapidly organize itself into a system of coherent, interacting vortices swimming through a sea of passive filamentary structure produced from earlier vortex interactions.

Note 2: The hydrostatic approximation $Dw/Dt \approx 0$ is valid when the vertical scale is much smaller than the horizontal scale of interest.

Note 3: The gradient wind balance can be achieved when the Rossby number is small, i.e., $(1/f)/(L/U) < 1$. The advective time scale is longer than the rotation time scale and thus the motion feels the existence of rotation.

Note 4: The angular momentum m is $1/2fr^2 + rv$, and the tangential wind v equation is the conservation of angular momentum.

On the Continuity Equation

Hung-Chi Kuo, 12/13/2007

The concepts of the state of a fluid may apply to a particular sample (or "parcel") that will move around when the fluid is in motion. Since nearby particles of fluid may move apart in time, it is necessary to think of an infinitesimally small sample that will retain its identity. This will called a *material element* of fluid (Batchelor, 1967, Chapter 2). The rate of change in the fluid state f following the material volume is denoted by $Df/Dt = \partial f/\partial t + \mathbf{V}\cdot\nabla f$ (the material derivative). The *material boundary* such as a free surface or an interface of two fluids is often encountered in the fluid dynamics. By its definition no fluid particles cross the material boundary, and thus a particle on the boundary will remain on the boundary. If $G(x, y, z, t) = 0$ is the equation of the boundary surface, G will always be zero for a material particle on this surface, and the therefore the material boundary condition in the fluid dynamics is

$$\frac{DG}{Dt} = 0.$$

[This condition is due to Lagrange (1781).]

In contrast with the situation of the fixed volume (control volume in the engineer term) V_0, the surface of a material volume V_m moves with the fluid.Let us the denote the surfaces of V_m and V_0 by ∂V_m and ∂V_0, and the velocity by $\mathbf{V}$, the mass conservation equation, the continuity equation, can be derived in the following ways.

(1) Material or Lagrange approach

With the fact that the material volume is squeezable and also with the divergent theorem, we have

$$\frac{D}{Dt}\int_{V_m} dV = \oint_{\partial V_m} \mathbf{V}\cdot\ d\mathbf{S} = \int_{V_m} (\nabla\cdot\mathbf{V})\, dV,$$

and the definition of $\nabla\cdot\mathbf{V}$ the rate of volume change following the fluid parcel,

$$\nabla\cdot\mathbf{V} = \lim_{\Delta\to 0}\frac{1}{\Delta V_m}\frac{D\Delta V_m}{Dt}.$$

Note that the squeeze of the material volume is denoted by $\oint_{\partial V_m}\mathbf{V}\cdot\ d\mathbf{S}$. The mass conservation within the material volume is

$$\frac{D}{Dt}\int_{V_m}\rho\, dV = 0,$$

and

$$\frac{D}{Dt}\int_{V_m}\rho\, dV = \int_{V_m}\frac{D\rho}{Dt}\, dV + \int_{V_m}\rho\nabla\cdot\mathbf{V}\, dV = 0,$$

(the Reynolds transport theorem) or

$$\int_{V_m}(\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{V})\, dV = 0.$$

Since the material volume V_m is arbitrary, we have

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{V} = 0.$$

The mass is conserved following the movement of a fluid parcel, and if $\nabla \cdot \mathbf{V} = 0$ that no volume changed, then we expect no density change following the movement of a fluid parcel. This is indeed as we see from the above equation that

$$\frac{D\rho}{Dt} = 0.$$

We call the fluid with $\nabla \cdot \mathbf{V} = 0$ incompressible fluid. A fluid of fixed density is an incompressible fluid. On the other hand, an incompressible fluid does not have to be of one fixed density.

(2) Fixed volume or control volume approach

The mass conservation in the material volume is

$$\frac{D}{Dt}\int_{V_0} \rho \, dV = -\oint_{\partial V_0} \mathbf{j} \cdot d\mathbf{S},$$

where the $\mathbf{j}$ is the flux and the right hand side the flux crossing the control surface. With the divergent theorem,

$$\int_{\partial V_0} \mathbf{j} \cdot d\mathbf{S} = \int_{V_0} (\nabla \cdot \mathbf{j}) \, dV.$$

$$\frac{D}{Dt}\int_{V_0} \rho \, dV = \int_{V_0} \frac{\partial}{\partial t}\rho \, dV = \int_{V_0} (\nabla \cdot \mathbf{j}) \, dV.$$

With the fixed volume arbitrary, we have the local conservation law

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0,$$

and with the flux $\mathbf{j} = \rho \mathbf{V}$ and we have the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0.$$

The continuity equation can be further approximated in the so called "anelastic" form. The anelastic continuity equation allows no acoustic waves in the system. The anelastic continuity equation can be derived from the continuity equation by setting $\rho = \rho_0(z)$. The vanish of $\partial \rho/\partial t$ can be achieved in the formal analysis by setting the time scale to the buoyancy time scale N^{-1}, where $N^2 = (g/\rho)(d\rho/dz)$ is the Brunt Vaisala frequency. With $\partial \rho/\partial t = 0$, we now have the anelastic continuity equation

$$\nabla \cdot (\rho_0 \mathbf{V}) = 0,$$

and in the cylindrical coordinate

$$\frac{\partial \rho_0 r u}{r \partial r} + \frac{\partial \rho_0 v}{r \partial \phi} + \frac{\partial \rho_0 w}{\partial z} = 0,$$

or

$$\frac{\partial ru}{r\partial r} + \frac{\partial v}{r\partial \phi} + \frac{1}{\rho_0}\frac{\partial \rho_0 w}{\partial z} = 0.$$

The equation can also be written as

$$\frac{\partial ru}{r\partial r} + \frac{\partial v}{r\partial \phi} + \frac{\partial w}{\partial z} + \frac{1}{\rho_0}\frac{d\rho_0}{dz}w = 0.$$

The last term is $-w/H$ when we have $\rho_0 = \rho(0)e^{-z/H}$. When shallow convection (i.e., the vertical scale is much smaller than the scale height H) is considered, the last term of above equation can be neglected. We then have the "incompressible" or "Boussineq approximation",

$$\frac{\partial ru}{r\partial r} + \frac{\partial v}{r\partial \phi} + \frac{\partial w}{\partial z} = 0.$$

Note that the continuity equation in general is $D\rho/Dt + \rho\nabla\cdot\mathbf{V} = 0$ and the Boussineq approximation is $\nabla\cdot\mathbf{V} = 0$. Thus, we can treat the $D\rho/Dt = 0$ equation in the incompressible or Boussinesq fluid as the thermodynamic equation. An incompressible fluid is just incompressible (unsqueezable and no material volume change), the density within the fluid can vary greatly and the buoyancy effect $-g\rho'/\rho$ can be very important. The Boussinesq or anelastic fluid allow no acoustic waves and thus simplified the multiple scale problem greatly. The incompressible fluid in Cartesian coordinate is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

An important question in the atmospheric and oceanic fluid dynamics is that in case of a significant $\partial u/\partial x$, what will be the dominant balance of the equation? In cases of strong rotation and strong internal stratification, the dominant balance of $\partial v/\partial y$ to the $\partial u/\partial x$ resulted in the horizontal quasi-nondivergent flow and the two-dimensional turbulence. The two-dimensional turbulence, which may be stemmed either from the fluid rotation or from the fluid internal stratification, has been a paradigm for geophysical fluid dynamics for many years. It is a striking fact that for any type of random initial state or external forcing, a two-dimensional fluid will rapidly organize itself into a system of coherent, interacting vortices swimming through a sea of passive filamentary structure produced from earlier vortex interactions.

On the Turbulence Kinetic Equation

Hung-Chi Kuo, 12/20/2007

With the Boussinesq approximation (incompressibility) in the Levi-Civita symbol,

$$\frac{\partial u_j}{\partial x_j} = 0,$$

the buoyancy is free from acoustic waves

$$-\frac{\rho}{\rho_0} = \frac{\theta}{\theta_0},$$

and thus hot (cold) air rises (sinks).

Let us denote $\overline{f}$ the Reynolds average of f, we have the momentum equations

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x_i} + \frac{g}{\theta_0}\theta\delta_{i3} + \nu\frac{\partial^2 u_i}{\partial x_j^2}, \tag{1}$$

and the averaged momentum equations

$$\frac{\partial \overline{u_i}}{\partial t} + \overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j} = -\frac{1}{\rho_0}\frac{\partial \overline{p}}{\partial x_i} + \frac{g}{\theta_0}\overline{\theta}\delta_{i3} - \frac{\partial}{\partial x_j}\left(\overline{u_i' u_j'}\right) + \nu\frac{\partial^2 \overline{u_i}}{\partial x_j^2}. \tag{2}$$

With the variable relationship $f = \overline{f} + f'$ and subtracting (2) to (1) we yield the perturbation equations

$$\frac{\partial u_i'}{\partial t} + \overline{u_j}\frac{\partial u_i'}{\partial x_j} + u_j'\frac{\partial \overline{u_i}}{\partial x_j} + u_j'\frac{\partial u_i'}{\partial x_j} - \frac{\partial}{\partial x_j}\left(\overline{u_i' u_j'}\right) = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x_i} + \frac{g}{\theta_0}\theta'\delta_{i3} + \nu\frac{\partial^2 u_i'}{\partial x_j^2}. \tag{3}$$

The turbulence kinetic energy equation can be achieved by the Reynolds average of $u_i' \times$ (3),

$$\frac{\partial K'}{\partial t} + \overline{u_j}\frac{\partial K'}{\partial x_j} = -\overline{u_i' u_j'}\frac{\partial \overline{u_i}}{\partial x_j} - \overline{u_i' u_j'\frac{\partial u_i'}{\partial x_j}} - \frac{1}{\rho_0}\frac{\overline{\partial p' u_i'}}{\partial x_i} + \frac{g}{\theta_0}\overline{w'\theta'} + \nu\overline{u_i'\frac{\partial^2 u_i'}{\partial x_j^2}}, \tag{4}$$

where $K' = \overline{u_i'^2}/2$ the turbulence kinetic energy. In the above derivation we have used the fact that

$$\overline{u_i'}\frac{\partial}{\partial x_j}\left(\overline{u_i' u_j'}\right) = 0.$$

The TKE equation can be written as (with the help of $\partial u_j'/\partial x_j = 0$)

$$\frac{\partial K'}{\partial t} + \overline{u_j}\frac{\partial K'}{\partial x_j} = -\overline{u_i' u_j'}\frac{\partial \overline{u_i}}{\partial x_j} - \frac{\partial}{\partial x_j}\left(\overline{\frac{u_i'^2 u_j'}{2} + \frac{p' u_i'}{\rho_0}}\right) + \frac{g}{\theta_0}\overline{w'\theta'} + \nu\overline{u_i'\frac{\partial^2 u_i'}{\partial x_j^2}}. \tag{5}$$

which can be expressed symbolically as follows:

$$\frac{\overline{D}K'}{Dt} = M + B + T - \epsilon,$$

where $\overline{D}/Dt$ is the rate of change following the mean motion, K' the turbulent kinetic energy per unit mass, M the mechanical production, B the buoyancy production or loss, T redistribution by transport and pressure forces (scale interactions by triple velocity and pressure velocity), and ϵ the dissipation of the smallest scales of turbulence by molecular viscosity. The TKE equation indicates that the TKE can be transported by the eddy, with the sources/sinks associated with the mechanical and thermal forcings, the internal scale exchanges, and the dissipation in the smallest scale.

Note 1: We define

$$\tau_{ij} = -\rho_0 \overline{u_i' u_j'}$$

the Reynolds stress. The Reynolds stress can be viewed as the u_i' momentum transported by the u_j' in the x_j directions.

With the mixing length hypothesis of Prandtle, a parcel of fluid that is displaced vertically will carry the mean properties of original level for a characteristic distance (the mixing length) and then mixed with its surroundings just as an average molecule travels a *mean free path* before colliding and exchanging momentum with another molecule. We can define

$$\tau_{ij} = \rho_0 K_m \left(\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i} \right)$$

where K_m is the eddy diffusivity. Note that the kinematic viscosity coefficient ν depends on the physical properties of the fluid while the eddy diffusivity K_m depends on the dynamics (flow motion) of the fluid. The averaged momentum equation can be expressed with the K_m and τ_{ij}

$$\frac{\partial \overline{u_i}}{\partial t} + \overline{u_j} \frac{\partial \overline{u_i}}{\partial x_j} = -\frac{1}{\rho_0} \frac{\partial \overline{p}}{\partial x_i} + \frac{g}{\theta_0} \overline{\theta} \delta_{i3} + K_m \frac{\partial^2 \overline{u_i}}{\partial x_j^2} + \nu \frac{\partial^2 \overline{u_i}}{\partial x_j^2}.$$

Energy removed by Reynolds stress is directly provide for turbulence while energy removed by viscous stress is directly dissipated, reappearing as heat.

Note 2: In a statically stable layer turbulence can exist only if mechanical production is large enough to overcome the damping effects of stability and viscous dissipation. This condition is measured by a quantity called the flux *Richardson number*,

$$Ri = -\frac{B}{M}.$$

Observations suggest that only when Ri is less than about 0.25 (i.e., mechanical production exceeds buoyancy damping by a factor 4) is the mechanical production intense enough to sustain turbulence in a stable fluid layer. A very simple theoretical argument for this value is presented in the Houze (1993, Cloud Dynamics book). His argument follows the consideration of two parcels of fluid, each of unit mass, initially separated by a small distance δz. The lower parcel is denser by an amount $\delta \rho = -(\partial \overline{\rho}/\partial z)\delta z$, where $\overline{\rho}$ is the mean state density of the fluid. If the parcels switch the position in the vertical, the increase of potential energy per unit volume of fluid is

$$g \, \delta\rho \, \delta z.$$

The only source of energy for this exchange is from the mechanical production associated with the shear of the mean flow $\partial\overline{u}/\partial z$. Let U_1 and U_2 be the initial velocities of the upper and lower parcels, respectively. The final velocity of the two parcels can be represented by $U_1 + \nabla$ and $U_2 - \triangle$, provided momentum is conserved in the exchange. The change in the kinetic energy per unit mass during the exchange is

$$(U_1 - U_2)\nabla + \triangle^2,$$

which is maximum for

$$\triangle = \frac{U_2 - U_1}{2}.$$

Let $U_2 = U$ and $U_1 = U + \delta U$ as well as the situation of maximum kinetic energy loss during the exchange, the kinetic energy loss per unit volume of fluid is approximately

$$\frac{\overline{\rho}(\delta U)^2}{4}.$$

From the $g\delta\rho\delta z$ and $\overline{\rho}(\delta U)^2/4$, it is clearly that for energy to be released in the exchange (in the form of an instability), we must have

$$g\ \delta\rho\ \delta z < \frac{1}{4}\overline{\rho}\ (\delta U)^2$$

or

$$Ri = \frac{-(g/\overline{\rho})(\partial\overline{\rho}/\partial z)}{(\partial\overline{u}/\partial z)^2} < \frac{1}{4},$$

here $\delta U = (\partial\overline{u}/\partial z)\delta z$. The vertical shear of the horizontal motion must be strong enough that the vorticity will be able to overcome the static stability.

Note 3: The basic properties of the Reynolds average are as follows:
(1) the average of the sum is the sum of the average;

$$\overline{f+g} = \overline{f} + \overline{g},$$

(2) constants do not affect and are not affected by averaging;

$$\overline{af} = a\overline{f},$$

where a is a constant, and $\overline{a} = a$. The above conditions imply that the average operator is "linear".
(3)the average of the time or space derivative of a quantity is equal to the corresponding derivative of the average;

$$\overline{\frac{\partial f}{\partial s}} = \frac{\partial\overline{f}}{\partial s},$$

(4) the average of the product of an average and a function is equal to the product of the average;

$$\overline{\overline{f}g} = \overline{f}\overline{g}.$$

Note that the running averages expressed by

$$\overline{f} = \frac{1}{2T}\int_{t-T}^{t+T} f(t')dt',$$

do not satisfy the last Reynolds conditions. The Grid-cell averages expressed by

$$\overline{f} = \frac{1}{2T}\int_{t_0-T}^{t_0+T} f(t')dt',$$

on the other hand, satisfy the last Reynolds condition. The draw back of the grid-cell averaged quantities are defined only at discrete points in space and time, and so are not spatially or temporally differentiable. In some applications this can be a disadvantage. The ensemble averages is an average over an infinite ensemble of realizations. It satisfies the Reynolds conditions, however, it goes without saying that turbulence data in practice is not averaged over an infinity of realizations. It is possible we use in a laboratory settings such as wind tunnel to produce the ensemble average or multiple simulations of the same situation in the high resolution model to produce the ensemble averages. It is difficult to produce the ensemble averages in the uncontrolled atmosphere and ocean.

Note 4: The typical buoyancy production in the atmosphere boundary layer can be estimated by

$$\frac{g}{\theta_0}\overline{w'\theta'} \sim \left(\frac{10ms^{-2}}{300K}\right)(1ms^{-1})(1K) \sim \frac{1}{30}\ m^2s^{-3}.$$

The typical heat driven boundary layer in the atmosphere is of 1km depth and it contains 1000 kg m^{-2} mass per unit area ($\rho_a \times D \sim 1$ kg $m^{-3} \times 1000$ m). The rate of energy production thus is

$$1000kgm^{-3} \times \frac{1}{30}m^2s^{-3} \sim 33\ Wm^{-2}.$$

The magnitude of the shear required in the mechanical production to have the similar effect as the buoyancy production can be estimated by

$$-\frac{d\overline{u}}{dz}\overline{u'w'} \sim \left(\frac{\triangle\,\overline{u}}{1000m}\right)(1ms^{-1})(1ms^{-1}) \sim \frac{1}{30}\ m^2s^{-3},$$

which gives $\triangle\,\overline{u} \sim 33$ ms^{-1}, namely, the wind speed at the top of 1km should be of the order of 33 m s^{-1}. The situation can be realized in the low level jet, or in the tropical cyclone. In general, the shear production can only be important locally while the buoyancy production may be important globally. The negative sign in the shear production can be understanded because $\overline{u'w'} < 0$ and $d\overline{u}/dz > 0$ in typical situations.

Vortex and Typhoon Dynamics

授課老師: 郭鴻基

作業及Take Home Exam.

規則

題目爲重要大氣流體力學課題複習, 內容爲我在 Purdue 大學教書時, 大氣動力學3次1小時考試的部分題目, 以及當時給學生的 Project Problems(含講義), 題目有挑戰性, 請及早動手。

---

1 (20 pts) Discuss the physical meaning of the following vector operations, also state whether the yield of vector operation is a scalar or a vector.

(a) $\nabla \cdot \mathbf{V}$

(b) $\nabla \times \mathbf{V}$

(c) $\nabla \phi$

(d) $\nabla^2 \phi$

(e) $\mathbf{A} \cdot \nabla \phi$

2 (20 pts) The approximate u component of equation can be written as

$$\frac{Du}{Dt} - \frac{uv \tan \phi}{a} = -\frac{1}{\rho} \frac{\partial p}{a \cos \phi \partial \lambda} + 2\Omega v \sin \phi.$$

(a) What are the characteristic magnitudes of each of the five terms for mid-latitude synoptic-scale motion?

(b) What is the Rossby number and what is its significance?

3 (30 pts) Consider the following equation [equation (1.27) in the text book] for transforming gradient quantities from height (z) to s vertical coordinates

$$\nabla_s f = \nabla_z f + \frac{\partial f}{\partial z} \nabla_s z,$$

where f is any arbitrary scalar parameter.

(a) Show that if $f = p$ and $s = p$,

$$g \nabla_p z = \frac{1}{\rho} \nabla_z p.$$

(b) Discuss the meaning of the above equation.

(c) What form of pressure gradient force take in the θ vertical coordinate?

4 (20 pts) An atmospheric with a dry adiabatic lapse rate (i.e. constant potential temperature) the geopotential height is given by

$$z = [1 - \left(\frac{p}{p_0}\right)^{R_d/c_p}] \frac{c_p \theta_0}{g}$$

where p_0 is the pressure at $z = 0$.

(a) What is the depth of this atmosphere?

(b) What is the temperature at the top of this atmosphere?

5 (20 pts) Consider the conservation of angular momentum $(\Omega a \cos\phi + u)a\cos\phi$ in a zonally symmetric, inviscid flow, compute the zonal wind of a parcel which has risen, at the equator, from the ocean surface to a height of 16 km above sea level and move poleward, at 16 km height, to ϕ N latitude. Assume the zonal velocity was zero at sea level. What is its zonal velocity at this latitude? Is the wind induced easterly or westerly?

6 (20 pts) Discuss what physical principles govern the dynamics of the atmosphere? [hint: conservation laws and constitutive equation]

7 (30 pts) Consider the continuity equation in the Eulerian form

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0.$$

(a) From the above expression derive the continuity equation in the Lagrange form

$$\frac{D\rho}{Dt} = -\rho[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}],$$

where

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}.$$

Discuss the meaning of continuity equation both in Eulerian and Lagrange forms.

(b) Why the term

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

has the meaning of rate of volume change? Please state your answer in mathematics.

(c) Explain why in the incompressible fluid (Boussinesq fluid) the density is conserved following the motion. Explain why the buoyancy force is often important in the incompressible fluid (Boussinesq fluid)?

8 (a) (20 pts) Derive and discuss the physical meaning of potential temperature based on the Poisson formula

$$\theta = T[\frac{p_s}{p}]^{R_d/c_p}.$$

(b) The potential temperature is conserved following the motion in the adiabatic flow. Why is the adiabatic flow relevant to the atmospheric/ocean fluid? Why do we need to define the potential temperature in the atmospheric sciences?

9 (30 pts) Let v be the tangential wind of a circular vortex with $v > 0$ ($v < 0$) denotes the counterclockwise (clockwise) flow. The centrifugal force in the circular vortex is v^2/r, the Coriolis force fv, and the pressure gradient force in the radial direction is $-1/\rho\partial p/\partial r$. The balanced flow equation for the circular vortex is

$$\frac{v^2}{r} + fv = \frac{1}{\rho}\frac{\partial p}{\partial r} = fv_g.$$

Discuss the following balanced flows; include your answer the balance of forces in mathematical form, force balance diagram, sign of flow (clockwise or counterclockwise or both), and a brief description of the flow.

(a) Geostrophic flow
(b) Cyclostrophic flow
(c) Inertial flow; what is the period of the inertial flow?
(d) Gradient flow (normal high)
(e) Gradient flow (normal low)
(f) In what fundamental way does gradient flow differ from (and similar to) geostrophic flow.

10 (30 pts) A good approximation to the tangential wind distribution in a tropical cyclone is the Rankine vortex

$$v(r) = \begin{cases} v_m\,(r/r_m)\,, & \text{for} \quad 0 \le r \le r_m; \\ v_m\,(r_m/r)\,, & \text{for} \quad r_m \le r \le \infty, \end{cases}$$

where r is the radius, v_m the maximum tangential wind, and r_m the radius of maximum tangential wind. The vorticity is given by

$$\zeta = \frac{\partial rv}{r\partial r} = \frac{v}{r} + \frac{\partial v}{\partial r}.$$

(a) Calculate the following as function of radius.

(i) relative vorticity
(ii) shear vorticity
(iii) curvature vorticity
(iv) circulation about a circle of radius r

(b) Sketch as a function of radius

(i) tangential wind
(ii) relative vorticity
(iii) circulation

11 (40 pts) The vector Ekman layer equation is

$$K\frac{\partial^2 \mathbf{V}}{\partial z^2} = f\mathbf{k} \times \mathbf{V} - f\mathbf{k} \times \mathbf{V}_g. \tag{1}$$

(a) What is K called? What is the typical magnitude for K in the atmosphere?

(b) What forces are represented in (1)?

(c) Assume the isobars are oriented in the x-direction and the wind vanishes at the surface. Draw the Ekman spiral solution for the wind field and the frictional force.

(d) To what dynamical field is the Ekman pumping proportional? Explain physically.

(e) Explain the terms "secondary circulations" and "spin-down".

(f) Use the equation (1) to prove that the total ageostrophic mass transport in the ocean is directed 90 degree to the right of the surface wind stress in the Northern Hemisphere. Mathematically this is to derive (3) from (2) from (1),

$$\mathbf{M}_a = \int_{-\infty}^{0} \rho(\mathbf{V} - \mathbf{V}_g)dz, \tag{2}$$

$$\mathbf{M}_a = -\frac{\rho K}{f}\mathbf{k} \times (\frac{\partial \mathbf{V}}{\partial z})_{z=0}. \tag{3}$$

(g) Give three phenomena that can be explained by (3).

12 (40 pts) The vector vorticity equation can be written as

$$\frac{\partial \boldsymbol{\zeta}}{\partial t} = -\mathbf{V}\cdot\nabla\boldsymbol{\zeta} - \boldsymbol{\zeta}(\nabla\cdot\mathbf{V}) - \boldsymbol{\zeta}\cdot\nabla\mathbf{V} + \nabla\times(-\frac{1}{\rho}\nabla p), \tag{4}$$

where $\boldsymbol{\zeta}$ is the vorticity vector.

(a) Write (4) in Levi-Civita symbol.

(b) Use the Levi-Civita symbol to derive the vorticity equation from the momentum equation

$$\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho}\nabla p. \tag{5}$$

(c) Discuss the meaning of each term in equation (4). Give 3 meteorological examples for the terms in equation (4).

(d) What is the difference between the relative vorticity and the absolute vorticity.

(e) Scale analysis of (4) in the mid-latitude synoptic scales.

13 (40 pts) Turbulence is of tremendous importance in both sciences and engineering. It is also one of the remaining big challenges in the physics. Concerning turbulence, Horace Lamb is quote in an address to the British Association for the advancement of Science as follows:

I am an old man now, and when I die and go to heaven there are two matters on which I hope for enlightment. One is quantum electrodynamics, and the other is the turbulence motion of fluid. About the former I am rather optimistic.

One of the basic equations in the turbulence is the TKE equation. The turbulence kinetic energy equation (e.g., (5.14) in Holton's text book) can be expressed symbolically as follows:

$$\frac{\overline{D}K'}{Dt} = M + B + T - \epsilon, \tag{6}$$

where $\overline{D}/Dt$ is the rate of change following the mean motion, K' the turbulent kinetic energy per unit mass, M the mechanical production, B the buoyancy production or loss, T redistribution by transport and pressure forces (scale interactions by triple velocity and pressure velocity), and ϵ the dissipation of the smallest scales of turbulence by molecular viscosity. The TKE equation indicates that the TKE can be transported by the eddy, with the sources/sinks associated with the mechanical and thermal forcings, the internal scale exchanges, and the dissipation in the smallest scale.

The TKE equation can be written as (with the help of $\partial u_j'/\partial x_j = 0$)

$$\frac{\partial K'}{\partial t} + \overline{u_j}\frac{\partial K'}{\partial x_j} = -\overline{u_i'u_j'}\frac{\partial \overline{u_i}}{\partial x_j} - \frac{\partial}{\partial x_j}\left(\overline{\frac{u_i'u_i'u_j'}{2} + \frac{p'u_i'}{\rho_0}}\right) + \frac{g}{\theta_0}\overline{w'\theta'} + \nu\overline{u_i'\frac{\partial^2 u_i'}{\partial x_j^2}}. \tag{7}$$

where $K' = \overline{u_i'u_i'}/2$ the turbulence kinetic energy.

(a) Derive the TKE equation (7) from the momentum equation (5) with the Reynolds average.

(b) Compare ((6) and (7) and discuss the meanings of each term.

(c) Discuss the situation that in a statically stable layer turbulence can exist only if mechanical production is large enough to overcome the damping effects of stability and viscous dissipation. What is the condition for instability?

14 (40 pts)

The potential vorticity concept is of vital importance in the advancement of atmospheric and oceanic dynamics in the last fifty years. The Ertel's potential vorticity is (also in the Levi-Civita symbol)

$$\mathrm{PV} = (\boldsymbol{\zeta} \cdot \nabla\theta)/\rho = \left(\zeta_i \frac{\partial \theta}{\partial x_i}\right)/\rho,$$

(a) Show that in a hydrostatic atmosphere ($\partial p/\partial z = -\rho g$), the potential vorticity can be written as

$$\mathrm{PV} = \frac{1}{\rho}\left(-\frac{\partial v}{\partial z}\frac{\partial \theta}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial \theta}{\partial y} + (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f)\frac{\partial \theta}{\partial z}\right). \tag{8}$$

(b) Show that in an isentropic vertical coordinate (i.e., θ vertical coordinate) the PV in (8) can be written as

$$\mathrm{PV} = \left((\frac{\partial v}{\partial x})_\theta - (\frac{\partial u}{\partial y})_\theta + f\right) / \left(-\frac{1}{g}\frac{\partial p}{\partial \theta}\right).$$

(c) What is meaning of the PV in isentropic vertical coordinate?

(d) Why do we need to define the PV instead of just using the vorticity? Discuss in general why about the word "potential" so special in physics? (hint: potential vorticity, potential temperature, potential density, and potential energy.)

15 (20 pts)

The text book presents the conservation principles both in the "advective form" (in the material derivative) and in the "flux form" (such as the conservation laws in the physics). Discuss the difference in both format of equations and give examples in atmospheric fluid.

16 (60 pts) The component equations of momentum in spherical coordinates (longitude λ, latitude ϕ, and vertical distance r)without the viscous term are

$$\frac{Du}{Dt} - (2\Omega + \frac{u}{r\cos\phi})(v\sin\phi - w\cos\phi) = -\frac{1}{\rho}\frac{\partial p}{r\cos\phi\partial\lambda}, \tag{9.1}$$

$$\frac{Dv}{Dt} + (2\Omega + \frac{u}{r\cos\phi})u\sin\phi + \frac{vw}{r} = -\frac{1}{\rho}\frac{\partial p}{r\partial\phi}, \tag{9.2}$$

$$\frac{Dw}{Dt} - (2\Omega + \frac{u}{r\cos\phi})u\cos\phi - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} - g, \tag{9.3}$$

where the material derivative in (9) can be expressed as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{u}{r\cos\phi}\frac{\partial}{\partial\lambda} + \frac{v}{r}\frac{\partial}{\partial\phi} + w\frac{\partial}{\partial r}.$$

The above equations are called the **nonhydrostatic primitive equations** or **the exact primitive equations**. They are not widely used at present. For large scale numerical weather prediction and general circulation modeling two approximations are made–**the traditional approximation** and **the quasi-static approximation**. The traditional approximation involves an approximation to the metric expression while the quasi-static approximation involves an approximation to the vertical equation. With the "traditional" approximation $z << a$(a is 6371 km, the earth radius), the above component equations can be written as

$$\frac{Du}{Dt} - \frac{uv\tan\phi}{a} = -\frac{1}{\rho}\frac{\partial p}{a\cos\phi\partial\lambda} + 2\Omega v\sin\phi, \tag{10.1}$$

$$\frac{Dv}{Dt} + \frac{u^2\tan\phi}{a} = -\frac{1}{\rho}\frac{\partial p}{a\partial\phi} - 2\Omega u\sin\phi, \tag{10.2}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g, \tag{10.3}$$

where the material derivative in (10) can be expressed as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{u}{a\cos\phi}\frac{\partial}{\partial\lambda} + \frac{v}{a}\frac{\partial}{\partial\phi} + w\frac{\partial}{\partial z}.$$

The quasi-static approximation involves the neglect of Dw/Dt in (10.3). When the Dw/Dt is omitted in (10.3), we have the "**quasi-static primitive equations**". The quasi-static primitive equations are widely used in the present NWP models and climate models.

(a) Prove that (9.1) can be written as the angular momentum principle

$$\frac{D}{Dt}[(\Omega r\cos\phi + u)r\cos\phi] = -\frac{1}{\rho}\frac{\partial p}{\partial\lambda}, \tag{11}$$

and (10.1) can be written as the angular momentum principle

$$\frac{D}{Dt}[(\Omega a\cos\phi + u)a\cos\phi] = -\frac{1}{\rho}\frac{\partial p}{\partial\lambda}. \tag{12}$$

Discuss how the conservation principle in (12) is a slightly distorted form of the exact principle (11).

(b) Derive the kinetic energy principle from the exact primitive equations (9.1)–(9.3) and the traditional approximation equations (10.1)–(10.3).

(c) Consider the formation of the subtropical jet in the Hadley circulation. The mean position of the subtropical jet is around 30 N and typical speed is around 60 $\sim$ 70 ms^{-1}. For zonally symmetric, inviscid flow, compute the zonal wind of a parcel which has risen, at the equator, from the ocean surface to a height of 16 km above sea level. Assume the zonal velocity was zero at sea level. Next assume this parcel moves poleward, at 16 km height, to 30 N. What is its zonal velocity at this latitude? Perform the calculations based on both (11) and (12). Since almost all present NWP models and general circulation models use the traditional approximation and thus have the approximate angular momentum principle (12), how large are the zonal wind errors in the jet due to the traditional approximation in such models? What other process(es) may make the observed jet speed smaller than the speed produced by the conservation of angular momentum?

(d) In an effort to gain more accuracy in the climate model, some reserachers have approximated (9.3) by the hydrostatic equation but have left (9.1) and (9.2) unchanged. Discuss why this is a bad idea.

(e) Some researchers have approximated (9.1)–(9.3) by simply replacing r by a (earth radius) (see Holton's book, page 35–38, equations (2.19)–(2.21)). Note that this is a different approximation than (9.1)–(9.3). Discuss why this is a bad idea.

(f) If a radius a spherical planet is covered with a homogeneous fluid with depath h and density ρ. Show that the surface pressure in hydrostatic balance is

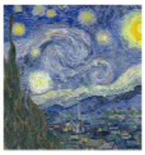
$$p = \rho g h \left(1 + \frac{h}{a} + \frac{1}{3}\frac{h^2}{a^2} \right).$$

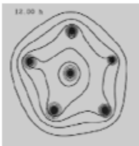
Note that the pressure is not the typical hydrostatic $\rho g h$ value. The atmospheric pressure or the geopotential height is often calculated with the hydrotatic balance, discuss the error involved when the spherical geometry of earth is considered.

第　十　六　屆　微　分　方　程　研　討　會

—演講題目—

兩度空間亂流與颱風動力





—主講人—

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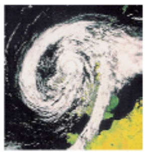
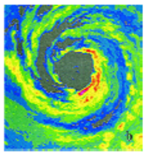
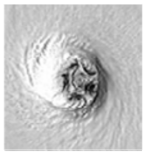
國立臺灣大學 大氣科學系

—時間—

九十七年元月四日

—地點—

交通大學

象神颱風侵襲後的台北市

平均一年至少**兩百億台幣**經濟影響

直接颱風災害
(風災、水災、土石流)

颱風放假與防災動員

水資源
(缺颱風雨來年必缺水)

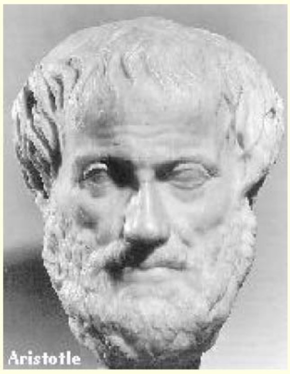
問世間颱風是何物？

- 夫大塊噫氣、其名為風 -- 莊子齊物論
- 「颱」-- 明末清初17世紀首見於漢文
- 康熙年間台灣府奏摺：今年亢旱之後、繼以颱風
  今歲入秋亢旱、繼又颶風大作
  台澎海外地方、每直秋潮節候、颱颶時有
- Typhoon -- 英文字根為16世紀阿拉伯語Touffon（旋轉）
- Typhon -- 希臘神話的風神
- Kamikaze -- 神風（？）13世紀末蒙古征日

渦旋，旋轉流體

Vortex，Rotational Dynamics

Aristotle's *Meteorologia*



Aristotle (384-322 BC) was a past master at asking questions.

He wrote the first book on Meteorology, the $Μετεωρολογια$ ($μετεωρον$: Something in the air)

This work dealt with the causes of various weather phenomena and with the origin of comets.

While a masterly speculator, Aristotle was a poor observer: for example, he believed that the lightning followed the thunder!

形而上學：憑藉第一原因，一切事物方能知曉，但其本身是自明的。

Sir Isaac Newton
(1642-1727)

Isaac Newton

Principia 1687

Nature and nature's law
lay hid in night,
God said,
Let Newton be,
and all was light.   A. Pope

Edmund Halley (1656–1742)

Edmund Halley was a contemporary and friend of Isaac Newton.

He was largely responsible for persuading Newton to publish his *Principia Mathematica*.

Halley and his Comet

Halley's analysis of what is now called Halley's comet is an excellent example of the scientific method in action.

A Tricky Question

If the Astronomers can make accurate 76-year forecasts, why can't the Meteorologists do the same?

- Size of the Problem 大氣海洋自由度無限 + 熱力學

  The solar system is discrete, with relatively few degrees of freedom; Dynamics is enough.

  The atmosphere is a continuum with (effectively) infinitely many variables; Thermodynamics is essential.

- Order versus Chaos 大氣海洋的混沌、蝴蝶效應

  The equations of the solar system are quasi-integrable and the motion is regular.

  The equations of the atmosphere are essentially nonlinear and the motion is chaotic.

Peter Lynch

Euler 18 century 流體動量、質量守恒

Fluid Dynamics

- Pressure gradient force 壓力梯度力

$$-\frac{1}{\rho}\nabla P$$

- Eulerian-Lagrangian transformation 座標轉換

$$\frac{d}{dt}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}$$

- Mass conservation (continuity equation) 質量守恒

$$\frac{\partial \rho}{\partial t}+\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}+\frac{\partial \rho w}{\partial z}=0$$

Isothermal sound speed (same mistake as Newton)

$$\left(\frac{\partial P}{\partial \rho}\right)_T \quad v.s. \quad \left(\frac{\partial P}{\partial \rho}\right)_\theta$$

Euler 1755

$$\frac{d}{dt}\int_{v_m}\rho\vec{v}\,dv=-\int_{\partial v_m}p\,d\vec{s}$$

$$\int_{v_m}\rho\frac{d\vec{v}}{dt}\,dv=-\int_{v_m}\nabla p\,dv$$

$$\rho\frac{d\vec{v}}{dt}=-\nabla p$$

Lagrange 1781

$$\frac{\partial \vec{u}}{\partial t}+\vec{\zeta}\times\vec{u}=-\frac{1}{\rho}\nabla p-\nabla K-\nabla\Phi$$

Rotation Vortex

Fig. 8.9. Vortices trailing form the wingtips of a Boeing 727. Figure courtesy of NASA.

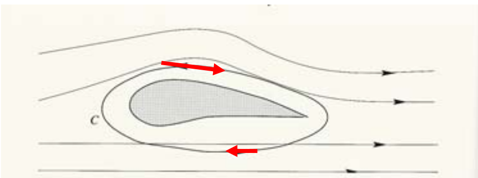


Fig. 8.10. Sketch of the flow along an airfoil. The wing is shown in grey, the contour C is shown by the thick solid line.

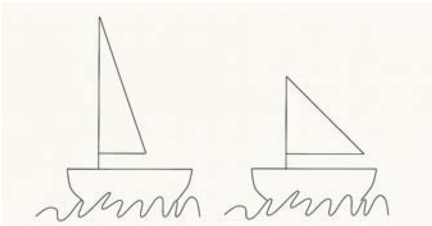


Fig. 8.12. Two boats carrying sails with very different aspect ratios.

Biomath

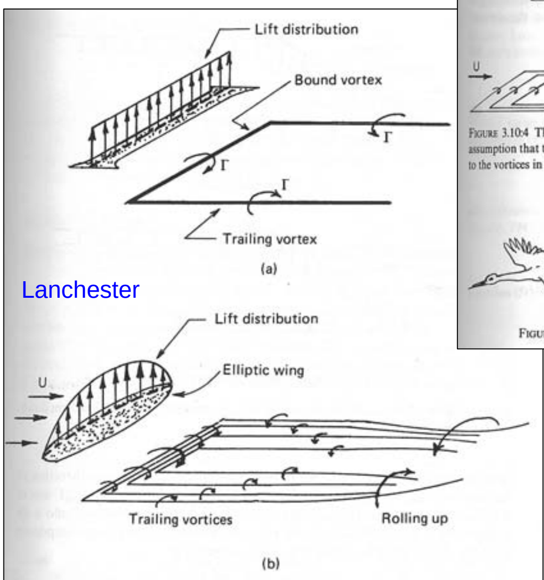


FIGURE 3.10:2 (a) A horse-shoe vortex representing a wing with a uniform lift distribution. (b) Lift distribution on an elliptic wing.

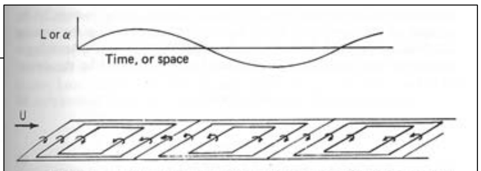


FIGURE 3.10:4 The vortices in the wake of an oscillating wing, idealized under the assumption that the lift fluctuation is very small so that the distortion of the wake due to the vortices in it is also very small. L is the lift, α is the angle of attack.

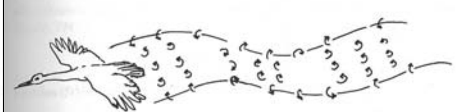


FIGURE 3.10:5 The vortex wake behind a stork in level flight.

Y. C. Fung

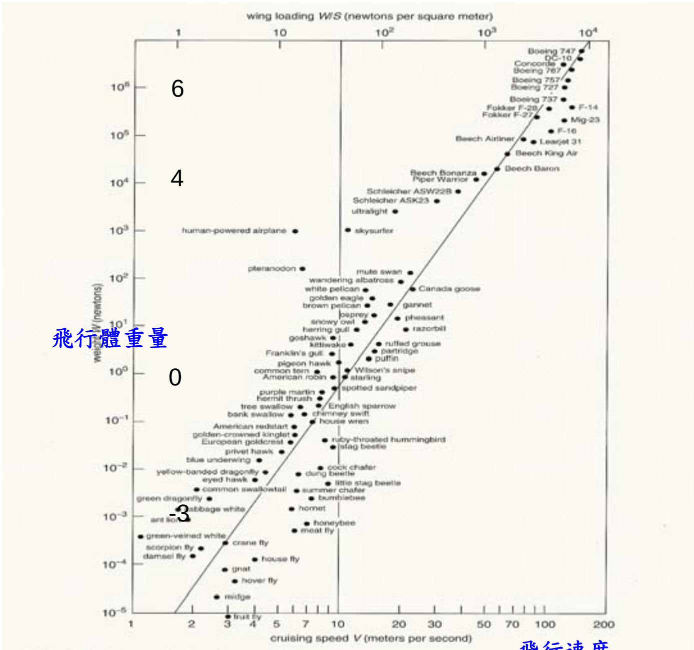


Fig. 2.2 The weight of many flying objects (vertical axis) against their cruising speed (horizontal axis) on a log–log plot. This figure is reproduced from reference [106] with permission from MIT Press.

Euler's Equations for Fluid Flow

Leonhard Euler, born on 15 April, 1707 in Basel. Died on 18 September, 1783 in St Petersburg.
Euler formulated the equations for incompressible, inviscid fluid flow:

$$\frac{\partial \mathbf{V}}{\partial t} + \boxed{\mathbf{V} \cdot \nabla \mathbf{V}} + \frac{1}{\rho} \nabla p = \mathbf{g} \,.$$

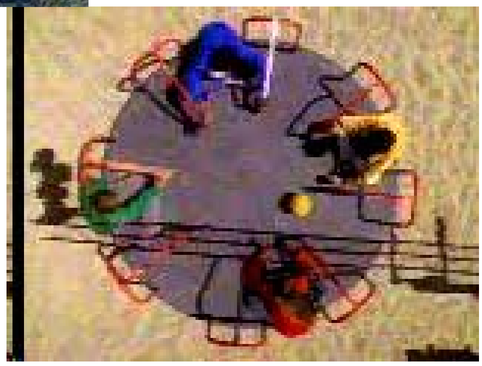
$$\nabla \cdot \mathbf{V} = 0$$

流體力學之父

Partial Differential Equations

偏微分方程式 **PDE**

非線性



Coriolis Force

Non-inertial Frame

fv: E-W Coriolis force

Conservation of angular momentum, 角動量守恒

Hadley 1735

旋轉力學

■ **fu: N-S Coriolis force**

Centrifugal force, thermal wind balance 向心力

Ferrel, 1859

■ **Coriolis force, Coriolis, 1835** 科氏力

▲ **Falkland ship battle in WW I**

■ **Laplace, (1740-1827)**

Atmospheric Observational net work (1800-1815)

Hydrostatic balance approximation

Tidal wave equation,

Laplacian

Adiabatic sound speed

Development of Thermodynamics 熱力學

19 century

第一定律 能量作功，能量守恆

First law: Energy is what makes it go and energy is conserved.

$\triangle Q = \triangle U + WORK$

Second law: Entropy tells it where to go!

第二定律 時間之矢，自然單向

Joule, Rudolf Clausius, Lord Kelvin and others

宏觀 微觀

Macro --- Micro

Classical and Statistical Thermodynamics

統計熱力學

Ludwig Boltzmann, 1844-1906, whose work led to an understanding of the macroscopic world on the basis of molecular dynamics.

S = k Log W

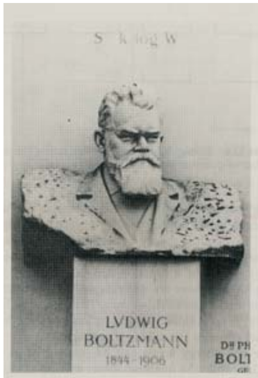
雲微物理

Precipitation

Enthalpy

Entropy

Gibbs Free energy



Ideal Gas Law Equation of State 理想氣體方程

- **1662, Boyle law, PV = c when T = c.**
- **1787, Charles law, V/T = c when P = c.**
- **1803, Gay-Lussac law, P/T = c when V = c.**
- **1811, Avagardro, 1 mole gas is 22.4 l in volume.**

Universal Gas Constant

$R^* = 8314.3$ J /(deg^{-1} kmol)

$PV = n R^* T$

$PV = m/M\ R^* T \quad P = m/V\ R^*/M\ T$

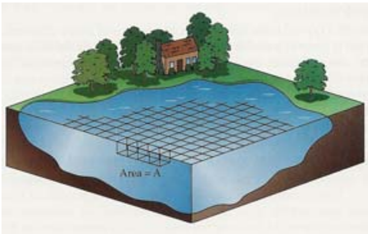
$P = \rho\ R T,\ R = R^*/M$

$R_d = 287$ J/deg.kg (R^*/M_d)

$R_v = 461$ J/deg.kg (R^*/M_v)

Estimate Avogadro's Number

Benjamin Franklin (1773)



Oil spreads on water
⟶ molecular size
⟶ Avogadro's number

(1) Molecular size

$$l = \frac{V}{A} = \frac{4.9\ \text{cm}^3}{2.0 \times 10^7\ \text{cm}^2} = 2.4 \times 10^{-7}\ \text{cm}$$

(2) Number of molecules

$$N = \frac{A}{l^2} = \frac{2.0 \times 10^7\ \text{cm}^2}{(2.4 \times 10^{-7}\ \text{cm})^2} = 3.5 \times 10^{20}\ \text{molecules}$$

(3) Mass of the oil

$$m = V \times D = 4.9\ \text{cm}^3 \times 0.95\ \frac{\text{g}}{\text{cm}^3} = 4.7\ \text{g}$$

(4) Number of moles of oil

$$\text{Moles of oil} = \frac{4.7\ \text{g}}{200\ \text{g/mol}} = 0.024\ \text{mol}$$

(5) Avogadro's number

$$\text{Avogadro's number} = \frac{3.5 \times 10^{20}\ \text{molecules}}{0.024\ \text{mol}} = 1.5 \times 10^{22}$$

Now we know: $N_A = 6.022142 \times 10^{23}\ /mol$

Brief History of Fluid Dynamics

Newton	**1700s**	Viscosity Law of motion for a particle	18世紀
Euler **D. Bernoulli**	**1750s**	Equations for inviscid flow, Law of motion applied to fluids	
Navier **Stokes**	**1827** **1845**	Equations for viscous fluid flow	19世紀
Boussinesq	**1877**	Turbulent mixing, eddy viscosity	
Reynolds	**1880**	Transition to turbulence, Reynolds number	
G. I. Taylor	**1915-1970**	Geophysical flows, rotating flows	20世紀
Prandtle	**1904**	Boundary Layer	

Planck, Unwilling Revolutionary: the idea of quantization

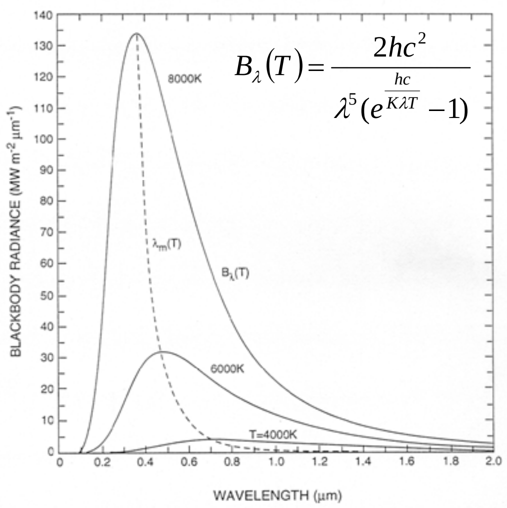


Figure 8.7 Spectra of emitted intensity $B_\lambda(T)$ for blackbodies at several temperatures, with wavelength of maximum emission $\lambda_m(T)$ indicated.

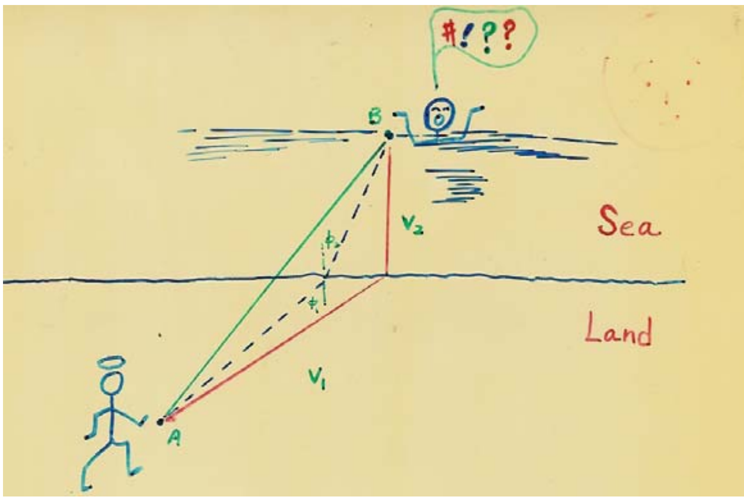
1900

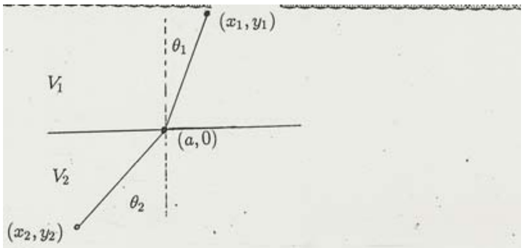
Hall of Fame in Science

Gravitational Law

Blackbody Radiation

E= MC^2





$$T(a) = \frac{[(x_1-a)^2+y_1^2]^{\frac{1}{2}}}{V_1} + \frac{[(x_2-a)^2+y_2^2]^{\frac{1}{2}}}{V_2}$$

$$\frac{\partial T}{\partial a} = 0$$

$$\frac{-(x_1-a)}{[(x_1-a)^2+y_1^2]^{\frac{1}{2}}}\frac{1}{V_1} + \frac{(a-x_2)}{[(x_2-a)^2+y_2^2]^{\frac{1}{2}}}\frac{1}{V_2} = 0$$

$$\boxed{\frac{\sin\theta_1}{\sin\theta_2} = \frac{V_1}{V_2}}$$

This is the Snell's Law.

一樣觀魚多樣情！

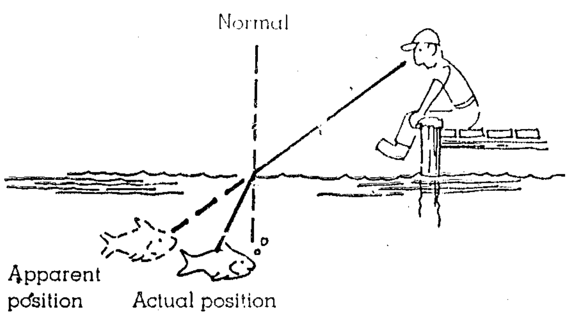


FIGURE 5.13 The refraction of light as it passes from the water into the less-dense air causes a fish to appear closer to the surface than it actually is.

（1）魚快樂嗎？
（2）熱血沸騰，立志革命！
（3）折射定律，最小原理。

Occam

Wikipedia

(1285～1349)

English Philosopher

Occam's Razor:

"What can be accounted for by fewer assumptions is explained in vain by more."

Vilhelm Bjerknes (1862–1951)

Peter Lynch

科氏力(18 19)

Momentum Conservation (18)

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}-fv=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\nabla^2 u$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+fu=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\nabla^2 v$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g+\nu\nabla^2 w$$

Mass conservation (18)

$$\frac{\partial \rho}{\partial t}+\frac{\partial u\rho}{\partial x}+\frac{\partial v\rho}{\partial y}+\frac{\partial w\rho}{\partial z}=0$$

Energy conservation (19)

$$\frac{\partial \theta}{\partial t}+u\frac{\partial \theta}{\partial x}+v\frac{\partial \theta}{\partial y}+w\frac{\partial \theta}{\partial z}=Q$$

Radiation 大氣輻射 (19,20)
Moisture Latent heat

Equation of State(17,18,19)

$$p=\rho R_a T,\qquad \theta=T\left(\frac{p_0}{p}\right)^{\frac{R_a}{C_p}}$$

(19)

雲物理 (19,20)

問蒼茫大氣，誰主浮沈？
質量、動量、能量與大氣狀態方程式

The Ultimate Problem in Meteorology Bjerknes 1911
氣象的終極問題

I The Present state of the atmosphere must be characterized as accurately as possible. 正確的觀測大氣現狀
[多重時空尺度]

II The intrinsic laws, according to which the subsequent states develop out of the preceding ones, must be known.
正確的大氣運作規律

Numerical Weather Prediction 數值天氣預報
[第一部電腦ENIAC, EBV model, 1950]
The Observation component 觀測
The diagnostic or analysis component 診斷分析
The prognostic component 預報

Lewis Fry Richardson, 1881–1953.

During WWI, Richardson computed by hand the pressure change at a single point.

It took him two years !

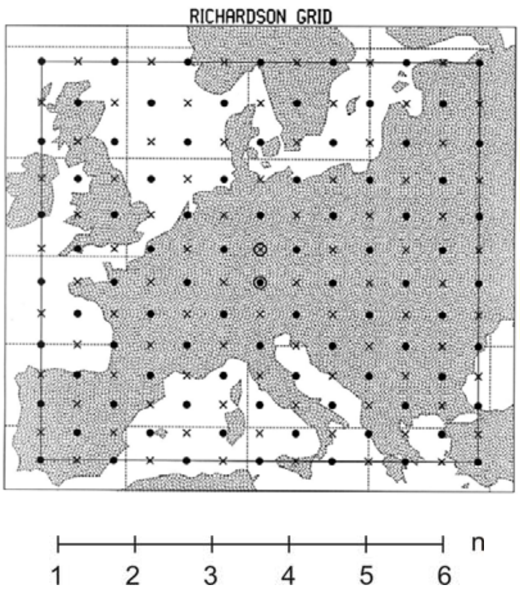
His 'forecast' was a catastrophic failure:

$$\Delta p = 145 \text{ hPa in 6 hours}$$

His method was unimpeachable.

So, *what went wrong?*

Peter Lynch



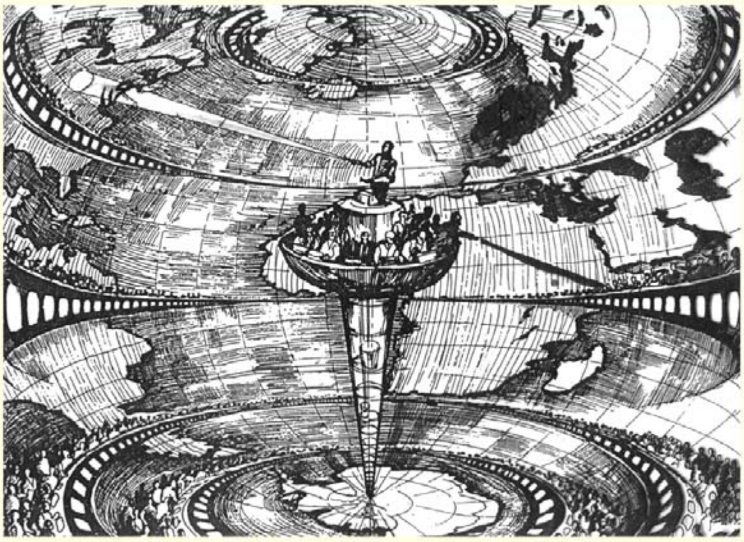
$$\frac{df}{dx} \rightarrow \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x}$$

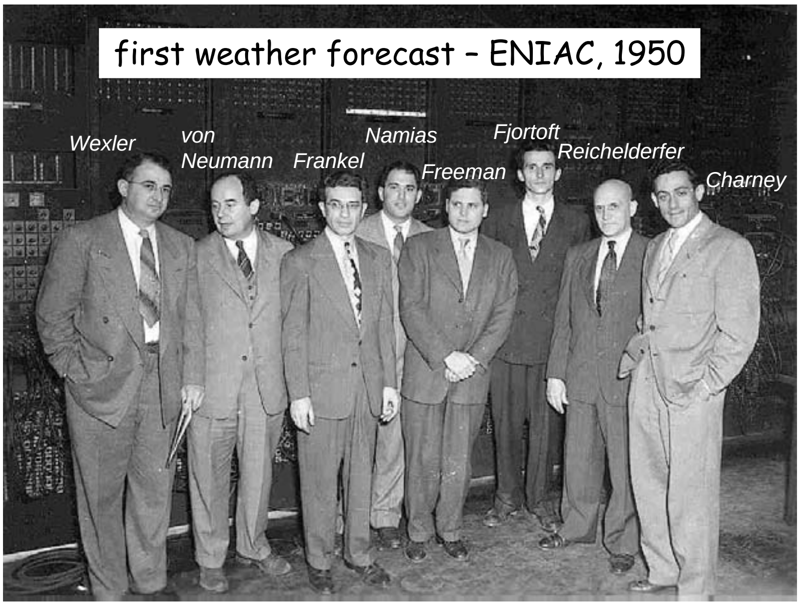
$$\frac{dQ}{dt} \rightarrow \frac{Q^{n+1} - Q^{n-1}}{2\Delta t} = F^{n}$$

13×13=169個ODE

169 自由度

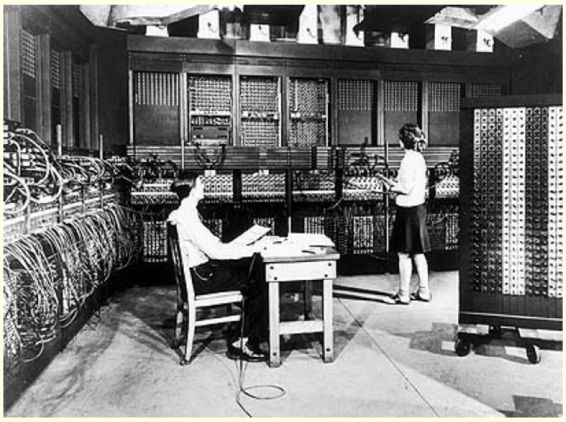
Richardson's Dream





In front of the Eniac, Aberdeen Proving Ground, April 4, 1950, on the occasion of the first numerical weather computations carried out with the aid of a high-speed computer.

The ENIAC Electronic Numerical Integrator and Computer



18000 vacuum tubes
70000 resistors
10000 capacitor
6000 switches

140 K Watts power

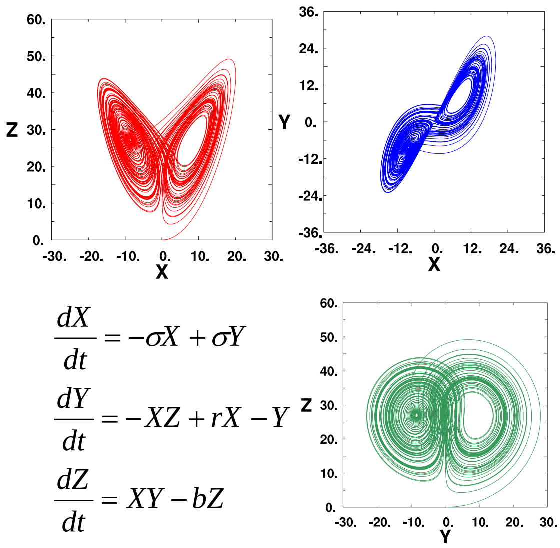
No high-level language
Assembly language

500 Flops
Function Table 0.001 s

3,700,000,000 times slower than current day large computer

第一部電腦　　氣象預報

Edward Norton Lorenz (1917～)　American Mathematician & Meterologist



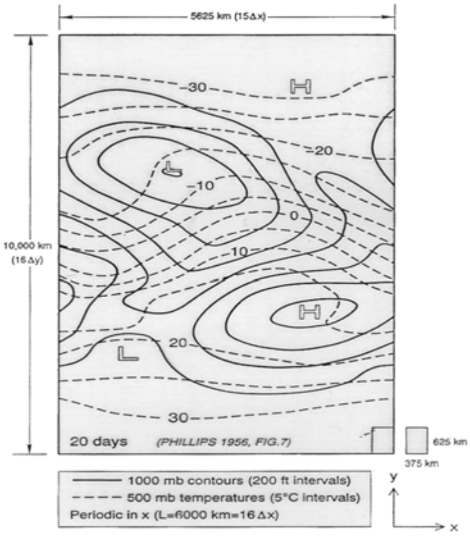
$$\frac{dX}{dt} = -\sigma X + \sigma Y$$

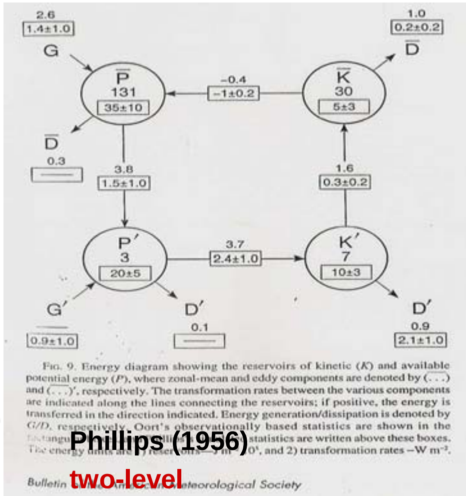
$$\frac{dY}{dt} = -XZ + rX - Y$$

$$\frac{dZ}{dt} = XY - bZ$$

$$\sigma = 10,\ r = 28,\ b = \frac{8}{3}$$

To study meteorology as an experimental science



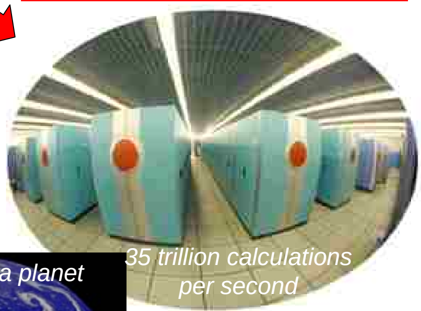


Phillips (1956)

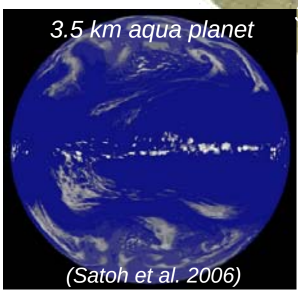
two-level Quasi-Geostrophic model

ENIAC – late 40s

Earth Simulator -- 2002



NASDA, JAERI, JAMSTEC



颱風潛熱與其它能量的比較

賀伯颱風的全台灣平均總雨量為400*mm*

400 *mm* = 0.4 *m*

$0.4\ m * 1000\ kg\ m^{-3} * 2.5\times10^{6}\ J\ kg^{-1}$

$= 10^{9}\ J\ m^{2}$

$10^{9}\ J\ m^{2} * 3.5\times10^{10}\ m^{2}$

$= 3.5\times10^{19}\ J \sim 10^{20} J$

$^{1}_{0}n + ^{235}_{92}U \rightarrow ^{142}_{56}Ba + ^{91}_{36}Kr + 3^{1}_{0}n$

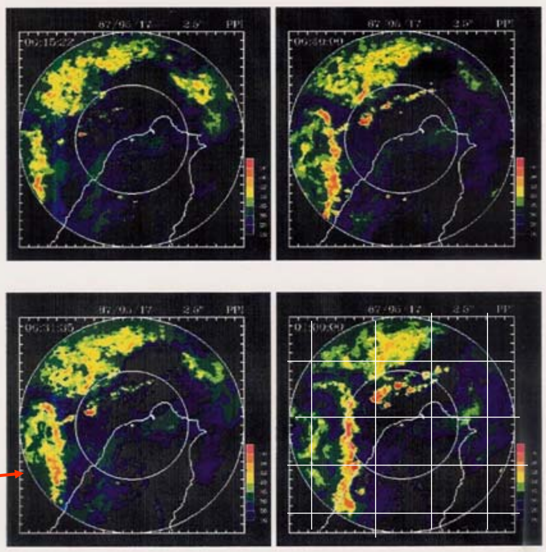
$1.68 * m * 10^{13} J/mol$

$\Rightarrow 1.46\times10^{6}\ kg\ U^{235}$ ($6*10^{6}$ mol)

能量估計值		備註
賀伯颱風降雨總潛熱能量	10^{20} J	可使台灣整層大氣增溫100度
台灣一年用電量	$5*10^{17}$ J	需數百年用電量才相當
全世界核子彈爆炸釋放能量	$2*10^{19}$ ~$2*10^{20}$ J	與賀伯颱風同等級
核戰後燃燒釋放能量	$2*10^{20}$ J	與賀伯颱風同等級
地球一天接受的太陽能量	$1.5*10^{22}$ J	數百個賀伯颱風
Tunguska隕石撞地球（西元1908年，西伯利亞）	10^{16} J	賀伯颱風的萬分之一
火流星撞地球（恐龍滅絕？）	$4*10^{23}$ J	數千個賀伯颱風

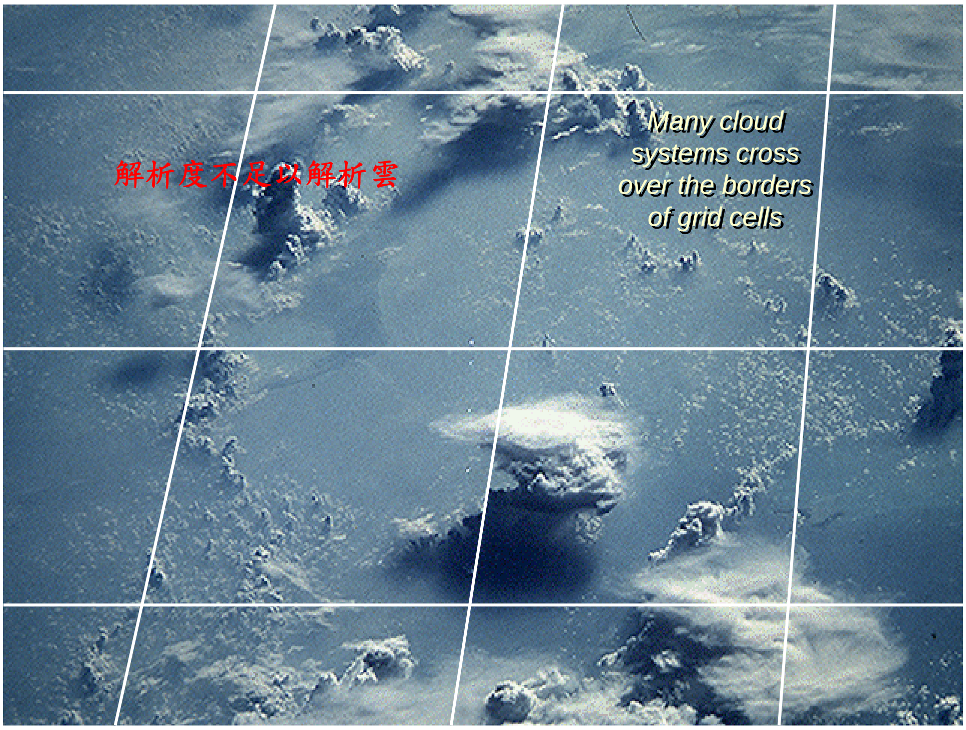
颮線

狂風不終朝
暴雨不終日
孰為此者
天地
天地尚不能久
而況人乎



Squall line

1987 年 5 月 17 日鋒前颮線之雷達回波圖



The Nature of Turbulence

Irregularity 不規則

Diffusivity 擴散

Large Reynold number 雷諾數大

Dissipation 損耗

Continuum (Multiple scales) 連續體（多重尺度）

Three dimensional vorticity fluctuations 三度空間
動能往小尺度

Rotation + Stratification 旋轉 ＋層化
2D turbulence ?? 兩度空間亂流？？
動能往大尺度

Leonardo da Vinci (1452-1519)

"Observe the movement of the surface of the water which resembles that of hair which has two motions, of which one depends on the weight of the hair and the other on the direction of the curls. Thus water forms eddying whirlpools of which one part depends on the predominant current and the other on the incidental motion and the return flow."

Multiple Scale Interactions
Steady and Turbulent Flows
多重尺度交互作用

Cascade (Nonlinear Dynamics)

So, nat'ralists observe, a flea
Hath smaller fleas that on him prey;
And these have smaller yet to bite 'em,
And so proceed *ad infinitum*.
--------- Jonathan Swift

Big whirls have little whirls
that feed on their velocity,
and little whirls have lesser whirls,
and so on, to viscosity.
----- Lewis Fry Richardson 1922

20th Century

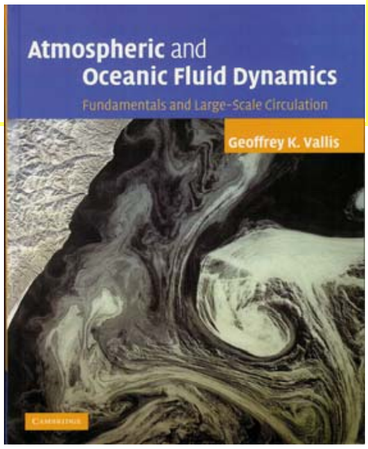
Geophysical Fluid Dynamics (GFD)

Atmospheric Oceanic Fluid Dynamics (AOFD)

is for those interested in doing research in the physics, chemistry, and/or biology of Earth fluid environment.

(b)

Fig. 9.2 Karman vortex streets in (a) the laboratory, for water flowing past a cylinder [From M. Van Dyke, *An Album of Fluid Motion*, Parabolic Press, Stanford, Calif. (1982) p. 56.], and (b) in the atmosphere, for a cumulus-topped boundary layer flowing past an island [NASA MODIS imagery].



Ertel's Derivation Potential Vorticity

$$\frac{\partial \zeta_i}{\partial t} + u_j \frac{\partial \zeta_i}{\partial x_j} + \zeta_i \frac{\partial u_j}{\partial x_j} = \zeta_j \frac{\partial u_i}{\partial x_j} + B_i$$ **Helmholtz Vorticity Equation (1858)**

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0$$ **Euler Continuity Equation (1750)**

$$\frac{d}{dt}\left(\frac{\zeta_i}{\rho}\right) = \frac{\zeta_j}{\rho}\frac{\partial u_i}{\partial x_j} + \frac{B_i}{\rho}$$

$$\frac{d}{dt}\psi = \dot{\psi}$$ **(some scalar function ψ)** **Thermodynamic equation**

$$\frac{d}{dt}\frac{\partial \psi}{\partial x_i} = -\frac{\partial u_j}{\partial x_i}\frac{\partial \psi}{\partial x_j} + \frac{\partial \dot{\psi}}{\partial x_i}$$

$$\frac{\zeta_i}{\rho}\frac{d}{dt}\frac{\partial \psi}{\partial x_i} = -\frac{\zeta_i}{\rho}\frac{\partial u_j}{\partial x_i}\frac{\partial \psi}{\partial x_j} + \frac{\zeta_i}{\rho}\frac{\partial \dot{\psi}}{\partial x_i}$$

+) $$\frac{\partial \psi}{\partial x_i}\frac{d}{dt}\left(\frac{\zeta_i}{\rho}\right) = \frac{\zeta_j}{\rho}\frac{\partial u_i}{\partial x_j}\frac{\partial \psi}{\partial x_i} + \frac{B_i}{\rho}\frac{\partial \psi}{\partial x_i}$$ $$\left(\frac{B_i}{\rho}\frac{\partial \psi}{\partial x_i} = \frac{1}{\rho}N(\rho, P, \psi)\right)$$

$$\frac{d}{dt}\left(\frac{\zeta_i}{\rho}\frac{\partial \psi}{\partial x_i}\right) = \frac{\zeta_i}{\rho}\frac{\partial \dot{\psi}}{\partial x_i} + \frac{1}{\rho}N(\rho, P, \psi)$$

Potential Vorticity

$$\frac{d}{dt}\int \vec{V}\cdot d\vec{l} = -\oint \frac{dp}{\rho}$$ **Kelvin theorem**

$$\frac{d}{dt}(\nabla\times\vec{V}\cdot\hat{n}\,\Delta A) = \vec{B}\cdot\hat{n}\,\Delta A \qquad \vec{B} = \nabla\times\left(-\frac{1}{\rho}\nabla p\right)$$ **Stokes' theorem**

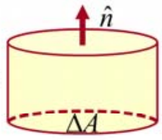
$$\frac{d}{dt}\left(\frac{\nabla\times\vec{V}\cdot\hat{n}}{\rho\,\Delta l}\right) = \frac{\vec{B}\cdot\hat{n}}{\rho\,\Delta l}$$

$$\frac{d}{dt}\left(\frac{\nabla\times\vec{V}\cdot\nabla\psi}{\rho}\right) = \frac{\nabla\times\vec{V}\cdot\nabla\psi}{\rho\,\Delta\psi}\frac{d\Delta\psi}{dt} + \frac{\vec{B}\cdot\nabla\psi}{\rho}$$

$$\frac{d}{dt}\left(\frac{\nabla\times\vec{V}\cdot\nabla\psi}{\rho}\right) = \frac{\nabla\times\vec{V}}{\rho}\cdot\nabla\frac{d\psi}{dt} + \frac{\vec{B}\cdot\nabla\psi}{\rho}$$

$$\frac{d}{dt}(\rho\,\Delta A\,\Delta l) = 0$$ **Euler's mass conservation**

$$\nabla\psi = \frac{\Delta\psi}{\Delta l}\hat{n}$$



Δl

$$\frac{\nabla\times\vec{V}\cdot\hat{n}}{\rho\,\Delta l} = \frac{\nabla\times\vec{V}\cdot\nabla\psi}{\rho\,\Delta\psi}$$

The conservation of PV is really the Kelvin theorem with a special but useful contour.

$$\frac{D\mathbf{V}}{Dt} + 2\mathbf{\Omega}\times\mathbf{V} = -\frac{\mathbf{1}}{\rho}\nabla_{\mathbf{z}}\mathbf{p} + \nu\nabla^2\mathbf{V}.$$

$$\frac{D\mathbf{V}}{Dt} + f\mathbf{k}\times\mathbf{V} = -\nabla_{\mathbf{p}}\phi + \nu\nabla^2\mathbf{V}.$$

Geostrophy
Rotation Dynamics

$$\epsilon\frac{D\mathbf{V}^*}{Dt^*} + \mathbf{k}\times\mathbf{V}^* = -\nabla^*_{\mathbf{p}}\phi^* + \frac{\epsilon}{\mathbf{Re}}\nabla^{*2}\mathbf{V}^*.$$

Singular Perturbation Problems
Quasi-balanced Dynamics

Boundary Layer Dynamics
Nearly Invicid

$$\epsilon = \frac{1/f}{L/U}$$ **Rotation time scale / Advective time scale**

$$Re = \frac{L^2/\nu}{L/U}$$ **Diffusion time scale / Advective time scale**

$$\frac{DP}{Dt} = \frac{\partial P}{\partial t} + u\frac{\partial P}{\partial x} + v\frac{\partial P}{\partial y} + w\frac{\partial P}{\partial z} \qquad P = \frac{\bar{\zeta} \cdot \nabla \theta}{\rho}$$

Potential Vorticity

↓ $\frac{\partial}{\partial z}$ $\quad P \to \zeta$

$$\frac{D\zeta}{Dt} = \frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y}$$

Barotropic Vorticity

↓

$$\frac{D\zeta_g}{Dt} = \frac{\partial \zeta_g}{\partial t} + u_g\frac{\partial \zeta_g}{\partial x} + v_g\frac{\partial \zeta_g}{\partial y}$$

$$\zeta_g = \nabla^2\psi \quad u_g = -\frac{\partial \psi}{\partial y} \quad v_g = \frac{\partial \psi}{\partial x}$$

Quasi-Geostrophic Vorticity

(Strong Rotation)

↓ $\nabla \cdot \vec{V} = 0$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial(\psi, \zeta)}{\partial(x, y)} = \nu\nabla^2\zeta$$

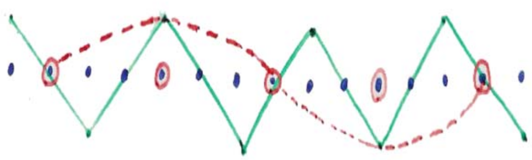
Nondivergent Barotropic Equation

Nonlinear computational instability
and **the Arakawa Jacobian (1966)**

$$J(\psi, \zeta)$$

➔**energy and enstrophy conservation**

➔**穩定氣候模式之數值計算的里程碑**



Nonlinear energy transfer
Aliasing error

2D Turbulence

Stratification and/or Rotation

Vortex Waves Turbulence

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} = \nu \nabla^2 \zeta$$

$$u = -\frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \psi}{\partial x}.$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial(\psi, \zeta)}{\partial(x, y)} = \nu \nabla^2 \zeta$$

Ocean Spice

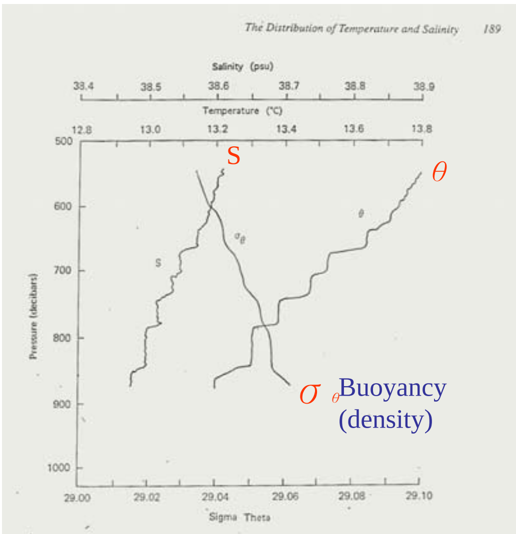


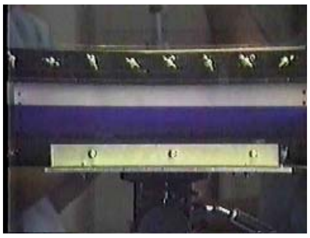
Figure 8.19 Temperature steps are usually concurrent with steps in the salinity gradient. The two tend to offset one another so that the density gradient is relatively smooth. (After Molcard and Williams, *Me. Soc. Roy. des Sci. de Leige, 7*, 1975.)

Some of the horizontal layering implied by discontinuities such as in Figure 8.19 can be explained on the basis of *stirring* (see "stirring and mixing" in Chapter 4). Because stirring in the horizontal plane is several orders of magnitude larger than that in the vertical, as different types of water mix horizontally, sharp vertical gradients can be expected to occur occasionally. A variety of additional mechanisms have been suggested for generating such fine structure, including turbulence generated by both bottom topography and surface waves as well as the breaking of small-scale internal waves (Figure 8.20).

Kelvin-Helmholtz Instability

Turbulent Mixing 晴空亂流

Intermediate wavelength



Multiple Scale Interactions in Vortex

Wave mean flow interaction in stable stratified fluid
Turbulent feed back to the vortex mean flow

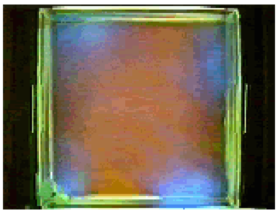
2D turbulence

Waves with zero potential vorticity

Non-rotation

rotation

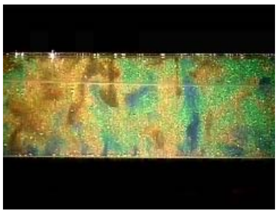
rotation



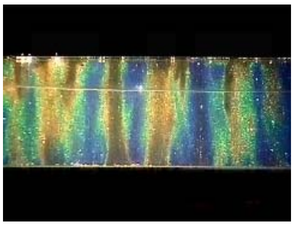
Gravity waves

Kelvin Waves
Edge waves

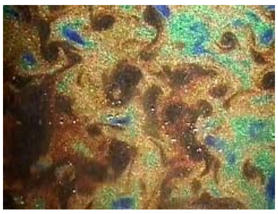
3D

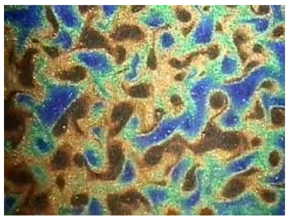


2D (strong rotation)



Taylor columns





Vortices with sharp edge

Kyoto Univ. GFD group

$$\frac{d}{dt}\int E(k)dk = 0. \qquad \frac{d}{dt}\left(\int k^2 E(k)dk\right) = \frac{d}{dt}\int Z(k)dk = 0.$$

$$\frac{d}{dt}\left(\int (k-k_1)^2 E(k)dk\right) > 0$$

$$\frac{d}{dt}\left(\int k^2 E(k)dk + k_1^2 \int E(k)dk - 2k_1 \int kE(k)dk\right) > 0$$

$$\frac{d}{dt}\left(\frac{\int kE(k)dk}{\int E(k)dk}\right) < 0,$$

Kinetic energy moves toward large scales

$$\frac{d}{dt}\left(\int (k^2-k_1^2)^2 E(k)dk\right) > 0$$

$$\frac{d}{dt}\left(\int k^2 Z(k)dk + k_1^4 \int E(k)dk - 2k_1^2 \int k^2 E(k)dk\right) > 0$$

$$\frac{d}{dt}\left(\frac{\int k^2 Z(k)dk}{\int Z(k)dk}\right) > 0,$$

Enstrophy moves toward small scales

Non-divergent barotropic model (Nearly Inviscid Fluid)

$$\frac{\partial}{\partial t}\zeta + J(\psi, \zeta) = \upsilon \nabla^2 \zeta \qquad \boxed{\nabla^2 \psi = \zeta}$$

The energy and enstrophy relations

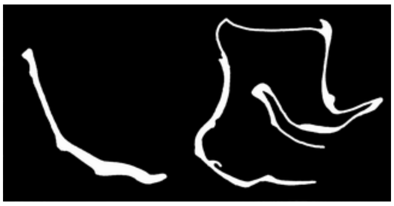
$$\frac{d\mathcal{E}}{dt} = -2\upsilon \mathcal{Z} \qquad \mathcal{E} = \iint \frac{1}{2}\left(u^2+v^2\right)dx\,dy \quad \text{kinetic energy}$$

$$\mathcal{Z} = \iint \frac{1}{2}\zeta^2\, dx\,dy \quad \text{enstrophy}$$

$$\frac{d\mathcal{Z}}{dt} = -2\upsilon \mathcal{P} \qquad \mathcal{P} = \iint \frac{1}{2}\nabla\zeta\cdot\nabla\zeta\, dx\,dy \quad \text{palinstrophy}$$

Batchelor 1969

Small viscosity led to large palinstrophy and the large enstrophy cascade



Stirring

A coffee lover's dream:

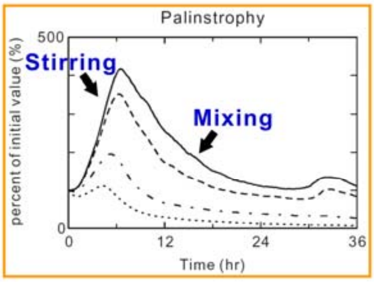
The best part of waking up, is the vortex in your cup!

$$\frac{D\theta}{Dt} = \frac{\partial\theta}{\partial t} + \vec{V}\cdot\nabla\theta = \nu\nabla^2\theta$$

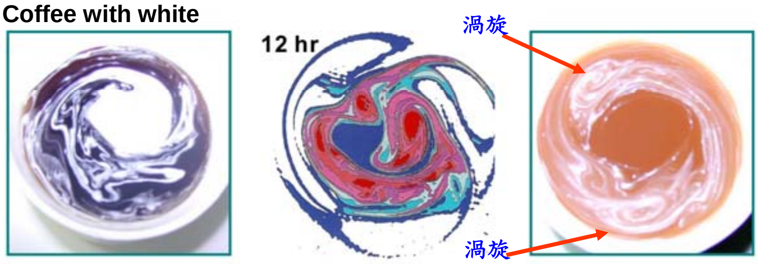
$$C = \frac{1}{2}\int \nabla\theta\cdot\nabla\theta \, dV$$

$$\frac{dC}{dt} = \int(\vec{V}\cdot\nabla\theta)\nabla^2\theta \, dV - \nu\int(\nabla^2\theta)\, dV$$

Stirring **Mixing**



Coffee with white



$E \sim p'^2 / L^2$ (KE) geostrophy

$Z \sim p'^2 / L^4$ (Enstrophy)

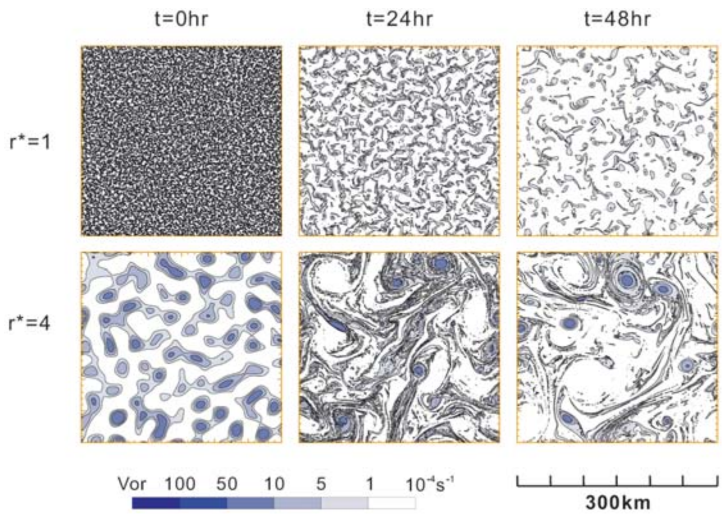
KE nearly conserved $L \sim p'$

Enstrophy cascade L↑ (L increase Z decrease)

Selective Decay of 2D turbulence

The vortices become, on the average,
larger, stronger, and fewer.

Merger and Axisymmetrization Dynamics



Fewer and stronger vortices !!!

小尺度變大尺度 **Coherent structure with filamentations in 2-D turbulence**

Weiss(1981,1991), Rozoff et al. (2004)

$$\frac{D}{Dt}(\nabla\zeta) = -J(\nabla\psi,\zeta)$$

$$\rightarrow \nabla\zeta(t) \propto \exp(\lambda t) \quad \lambda = \pm\frac{1}{2}\sqrt{Q} = \pm\frac{1}{2}\sqrt{S_1{}^2 + S_2{}^2 - \zeta^2}$$

$$S_1 = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \quad \text{(stretch deformation)}$$

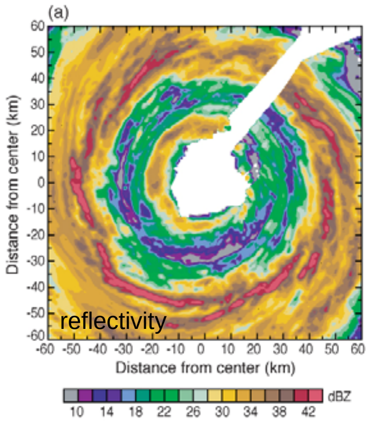
$$S_2 = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \quad \text{(shear deformation)}$$

Q > 0 (strain dominates)
→vorticity gradient will be stretched
Q < 0 (vorticity dominates)
→vortex is stable (survival of eyewall meso-vortices)

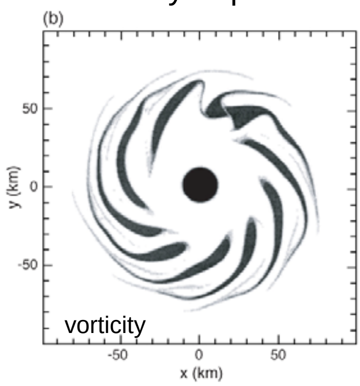
RAINEX (2005)

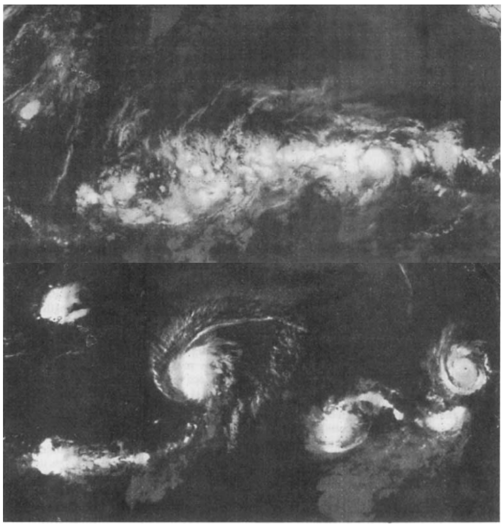
Shear/strech deformation outside the radius of maximum wind

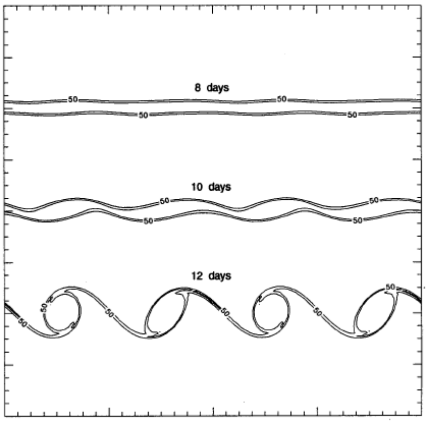
ELDORA composite
~120 x 120 km^2



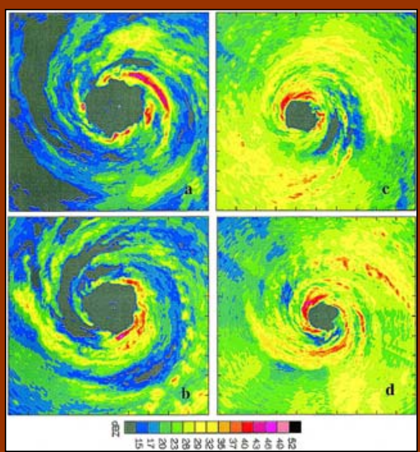
Kossin et al. (2000)
Instability Experiment



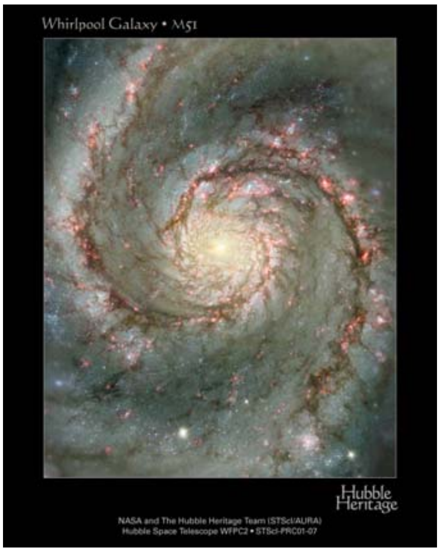




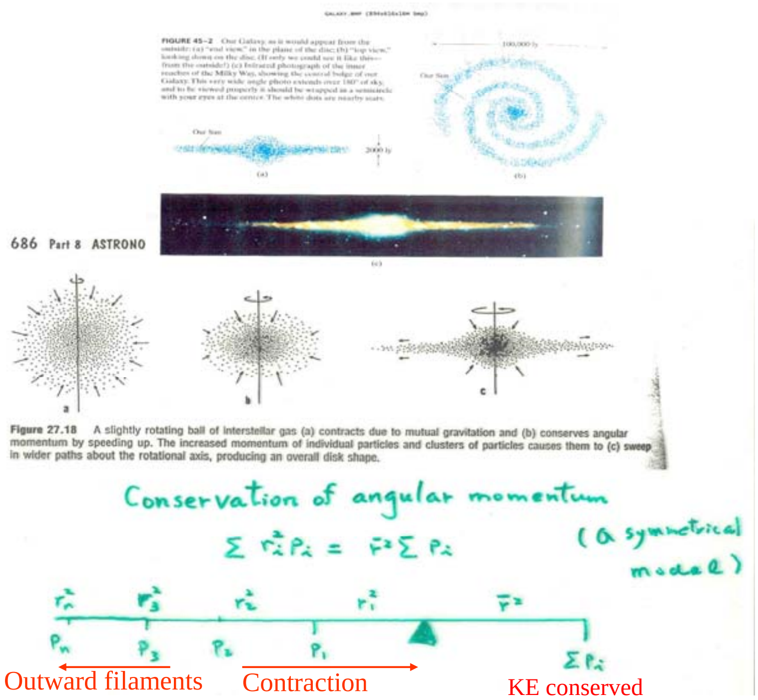
Spiral Band in Hurricane and Galaxy



Airborne-radar reflectivity in Hurricanes
Guillermo (1997) (left panels) and Bret (1999) (right panels).



Kossin and Schubert 2001

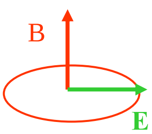


Electron density redistribution in experimental plasma physics

single sign charge
+
axial magnetic field
confinement

$$\mathrm{E} = -\nabla\psi$$

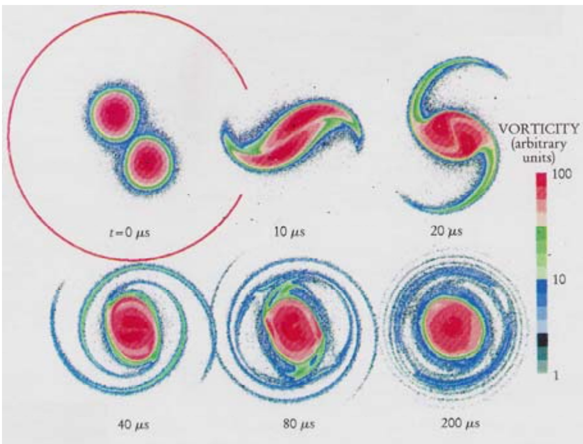
$$\nabla \cdot \mathrm{E} = -\nabla^2\psi = \frac{\rho}{\epsilon}$$



$\vec{E} \times \vec{B}$ drift

Coriolis force

Axisymmetrization 軸對稱化



Core is protected, thin filaments from edges

Bowmen and Mangus (1993)

臭氧洞衛星觀測

Observations of deformation and mixing of the total ozone field in the Antarctic polar vortex

核心空氣被渦旋鎖住

細絲帶

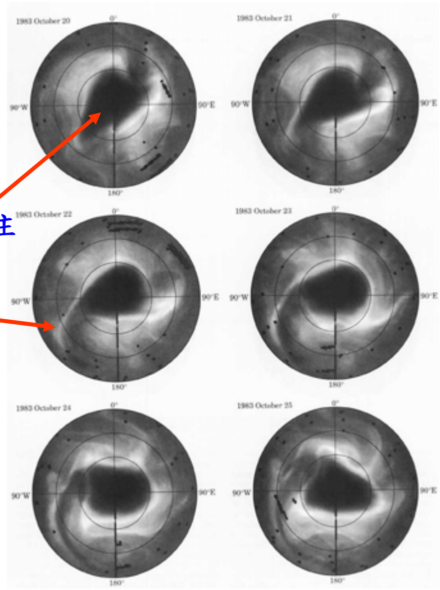


Fig.1: Daily TOMS images of total ozone in the Southern Hemisphere for six consecutive days in October 1983. Latitude circles are drawn at 40˚,60˚, and 80 ˚S. The outermost latitude is 20 ˚S.

Huang and Robinson 1998

Forcing added

Inverse energy cascade to zonal Harmonics

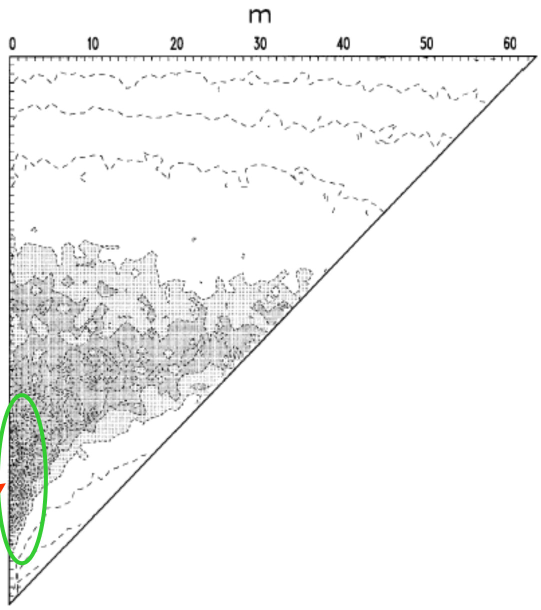


FIG. 2. Energy spectrum $E(m, n)$ of an ensemble mean at day 80 of 10 decaying turbulence experiments. The magnitude of the spectrum is normalized by the maximum value on the map. Contour levels are 0.0001, 0.001, 0.01, 0.1–0.9 with increment 0.1. Area with $E(m, n) > 0.1$ is lightly shaded, $E(m, n) > 0.2$ heavily shaded.

9.84 hr rotating period
Jupiter

Huang and Robinson 1998

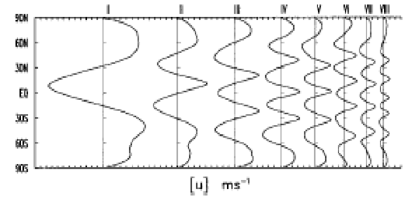
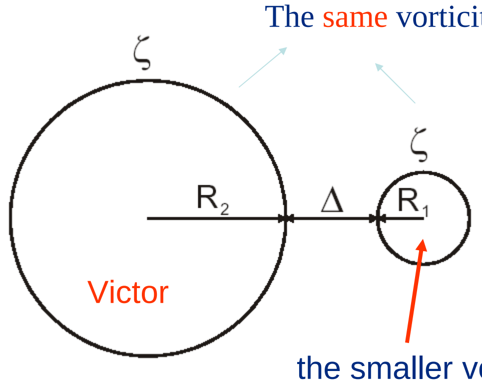


FIG. 4. Time-mean zonal-mean zonal wind profiles for cases I–VIII in Table 1 (the eight open circles in Fig. 3). Each grid on the abscissa represents 1 m s^{-1}.

These alternating Easterly and westerly Jets are similar to observed patterns on Jupiter and Saturn.

Dritschel and Waugh (1992)

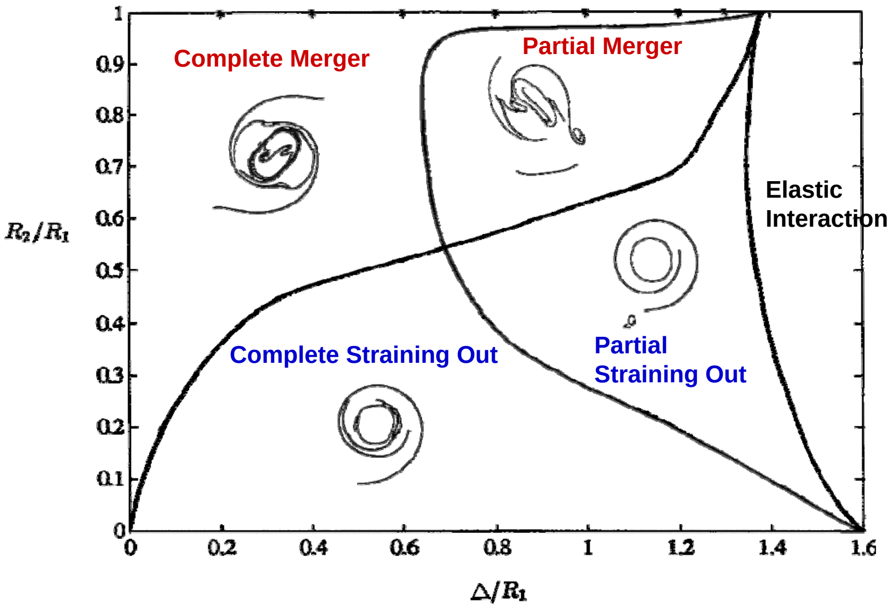
The same vorticity strength



the smaller vortex was the one often being partially or totally destroyed

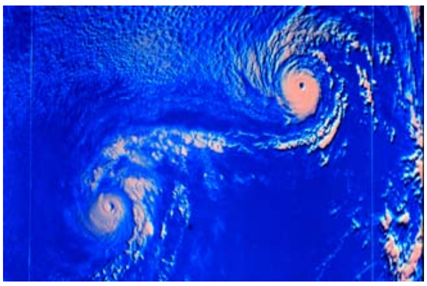
$$r = \frac{R_1}{R_2}$$ Vortex radius ratio

$$\Delta / R_1$$ Dimensionless gap

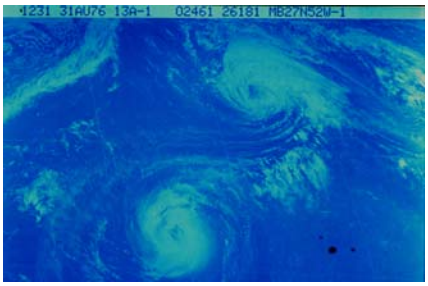


(Adapted from Dritschel and Waugh 1992.)

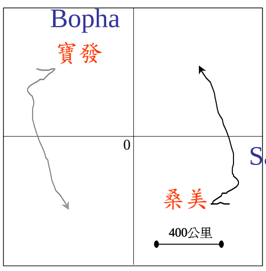
雙颱風的互繞 ----藤原效應



颱風 Ione 與 Kristen



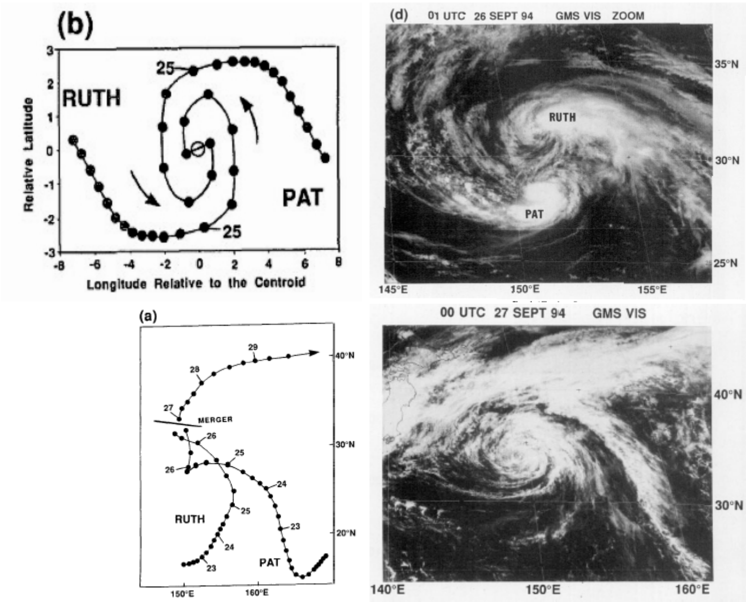
颱風 Emmy 與 Frances



The unusual south movement of Typhoon Bopha

Merger

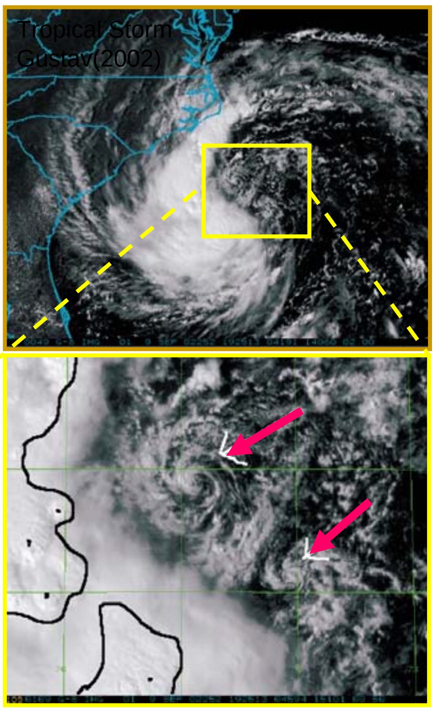
雙颱風互繞（藤原效應）合併 ---- 颱風 PAT 與 RUTH

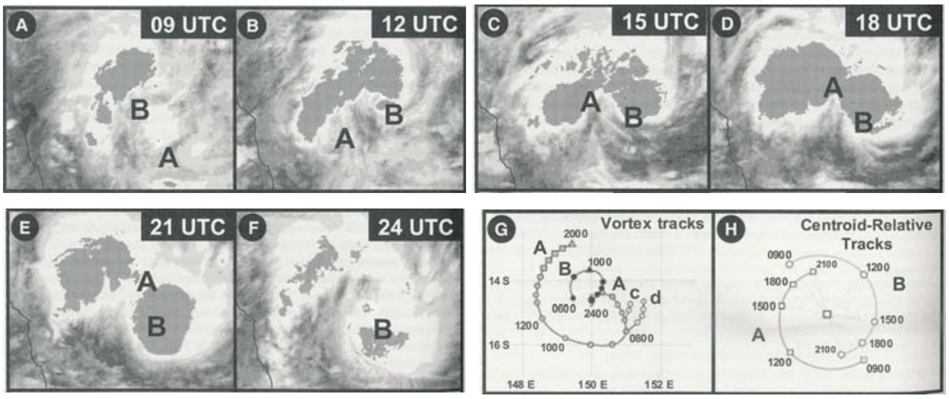


Hendricks et al. (2004)

"Vortical" Hot Towers

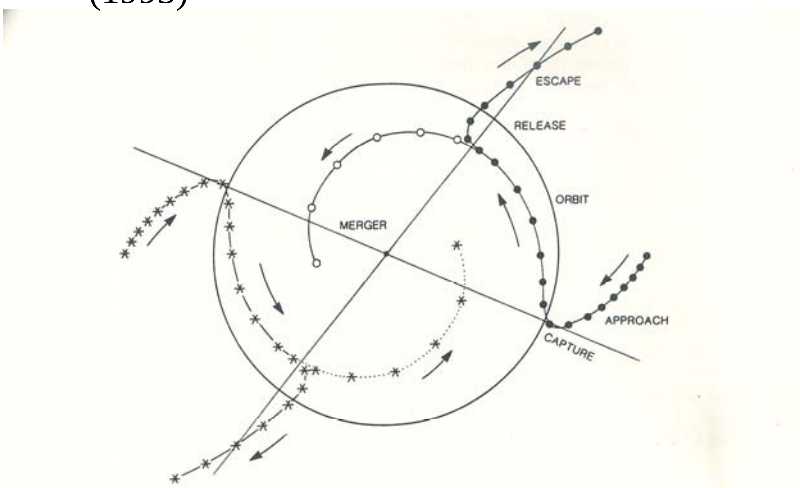
- **The net effect of the hot towers is to produce strong small-scale (10km in diameter on average) lower-tropospheric (below z ≈ 5km) cyclonic PV towers.**
- **The strong updrafts in the hot towers converge and stretch existing low-level vertical vorticity into intense small-scale vortex tubes.**
- **Multiple mergers / axisymmetrization of these tubes in the lower troposphere.**





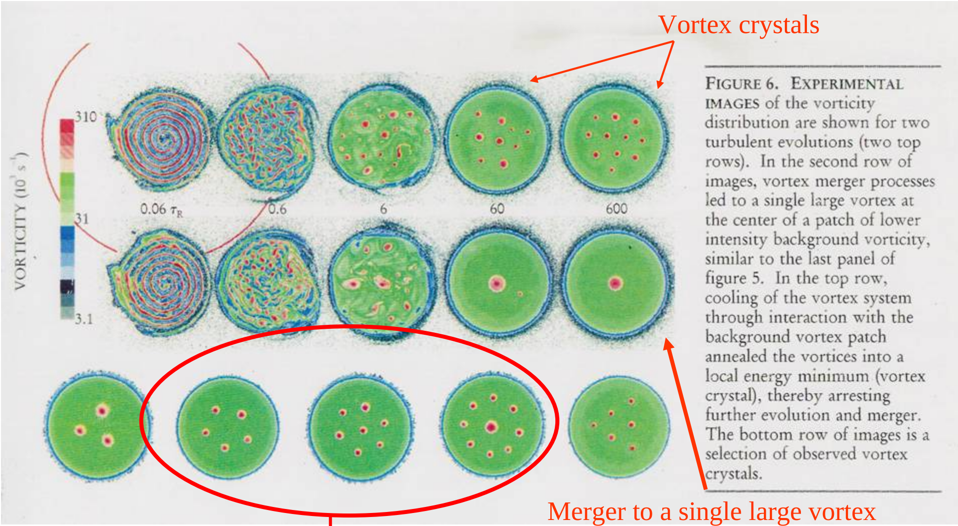
(A)-(F) The locations of the two mesovortices A and B during the development of Tropical Cyclone Oliver superposed on three-hourly satellite imagery for the period 0900 UTC February 4 to 0000 UTC February 5, 1993; (G) Tracks of four of the vortices obtained from radar data. The positions are not evenly spaced and so times (in UTC) of some of the vortex positions are marked; (H) three-hourly centroid-relative tracks of mesovortices A and B from 0900 UTC to 2100 UTC February 4[Simpson et al., 1997].

Lander and Holland (1993)



Merger and Elastic Interaction **Chaos**

Two Turbulent Evolutions in single sign charge plasma



Coherent Structure 調理分明結構

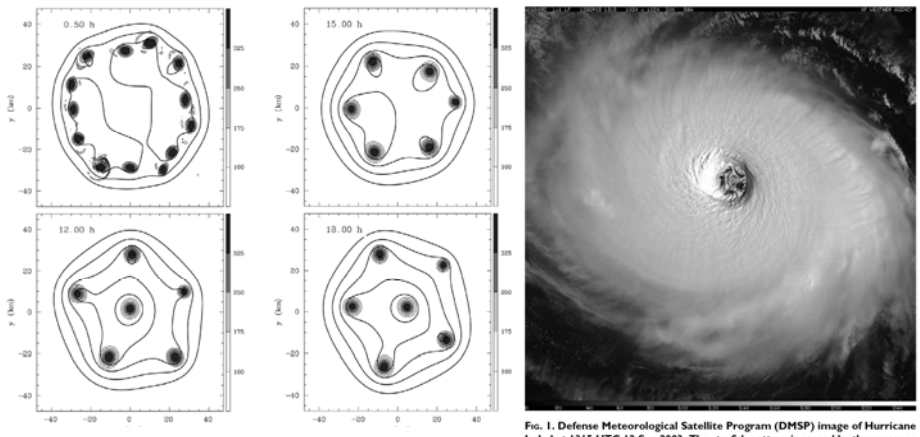


FIG. 2. Evolution of vorticity (shaded) and streamfunction contours (bold) for the numerical experiment of Kossin and Schubert (2001). Values along the label bar are in units of 10^{-4} s^{-1}. The shape of the streamlines transitions from a pentagon to a hexagon and back to a pentagon over 6 h.

FIG. 1. Defense Meteorological Satellite Program (DMSP) image of Hurricane Isabel at 1315 UTC 12 Sep 2003. The starfish pattern is caused by the presence of six mesovortices in the eye—one at the eye center and five surrounding it.

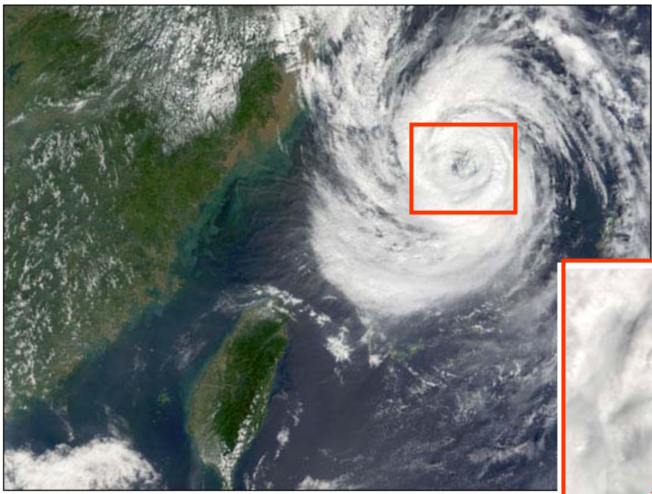
MESOVORTICES IN HURRICANE ISABEL

BY JAMES P. KOSSIN AND WAYNE H. SCHUBERT

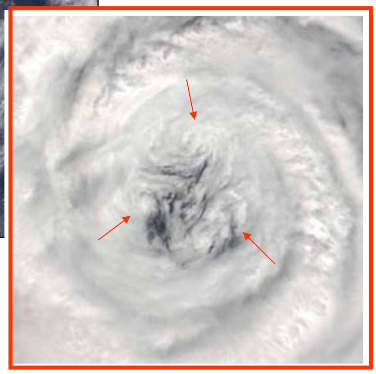
Importance of Asymmetric Vorticity Dynamics

A case where MPI theory failed! (Montgomery 2006)

納莉颱風眼附近的中尺度渦旋

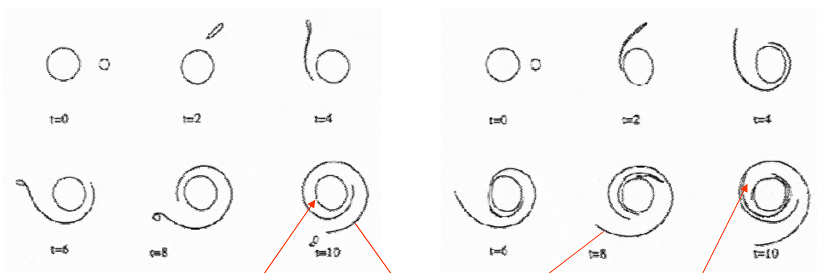


侵台納莉颱風登陸前
颱風眼附近觀測到3 個
中尺度渦旋



Straining out regime

partial straining - out (PSO) complete straining - out (CSO)



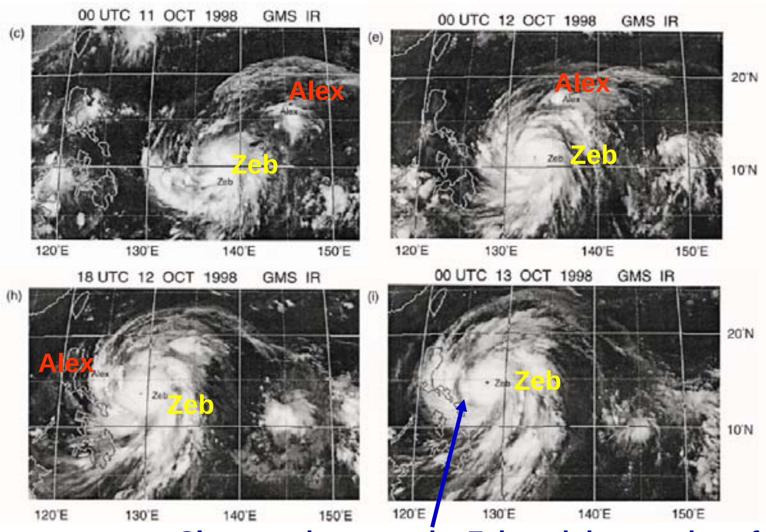
Clear gap Clear gap

Adverse shear effect

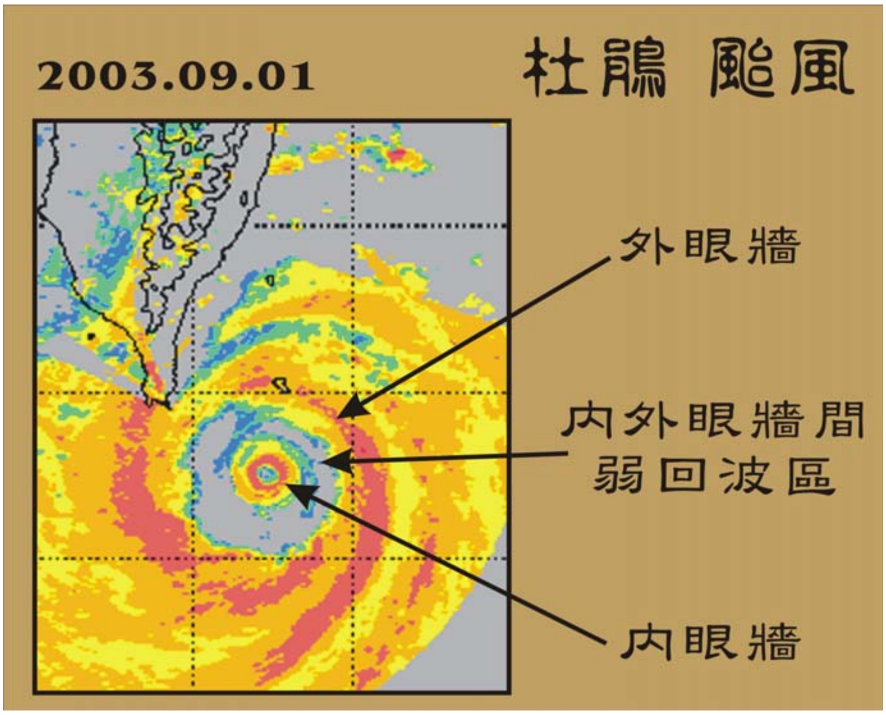
The bands are too thin to be called concentric eyewalls.

颱風渦旋合併動力探討研究

Kuo, H.-C., G. T.-J. Chen, and C.-H. Lin, 2000: Merging processes of tropical cyclone Zeb and Alex. *Mon. Wea. Rev.*, **128**, 2967-2975.

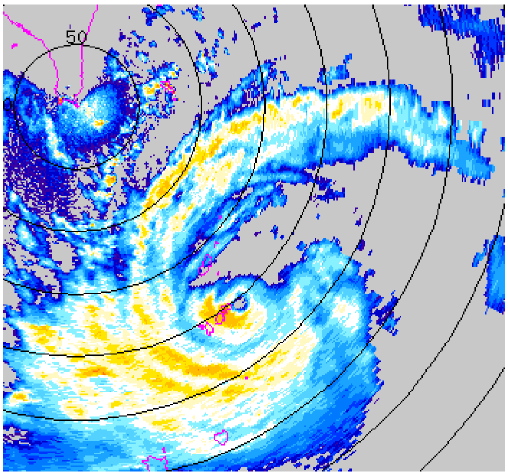


Clear gap between the Zeb and the remains of Alex

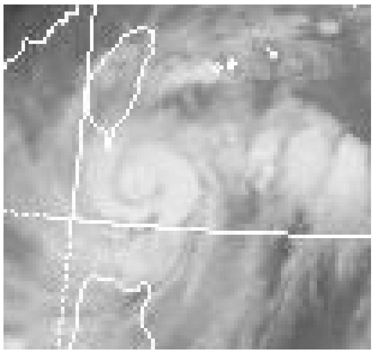


The Formation of Concentric Vortex Structure in Typhoons

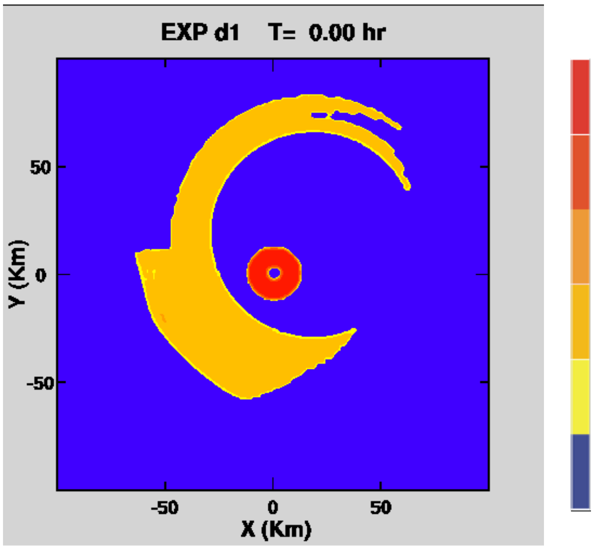
Typhoon Lekima (2001) 0935-1935 LST



0925 1900LST



Formation of concentric vorticity structure in Lekima (2001)

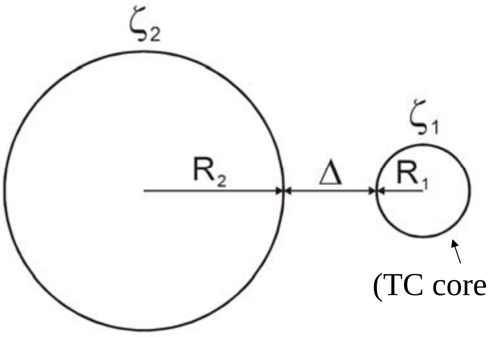


Binary vortex interaction

$$r = \frac{R_1}{R_2}$$

$$\Delta / R_1$$

$$\gamma = \frac{\zeta_1}{\zeta_2}$$



An extension of Dritschel and Waugh's (1992) work.

In addition to the radii ratio and the normalized distance between the two vortices, the vorticity ratio is added as a third external parameters.

Interaction of a small and strong vortex (representing TC core) with a large and weak vortex (representing vorticity induced by convection outside the TC core)

$\gamma = 10$

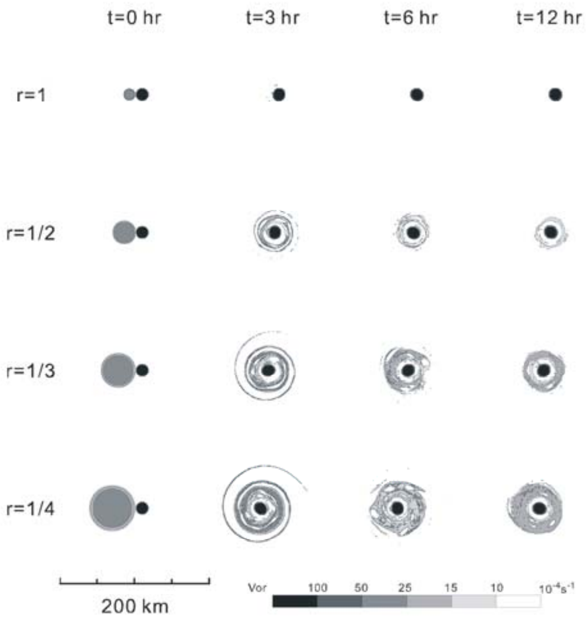
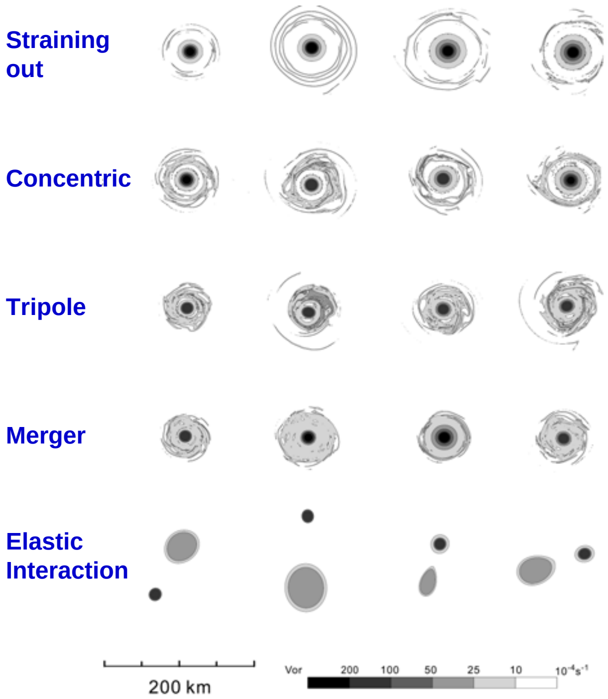
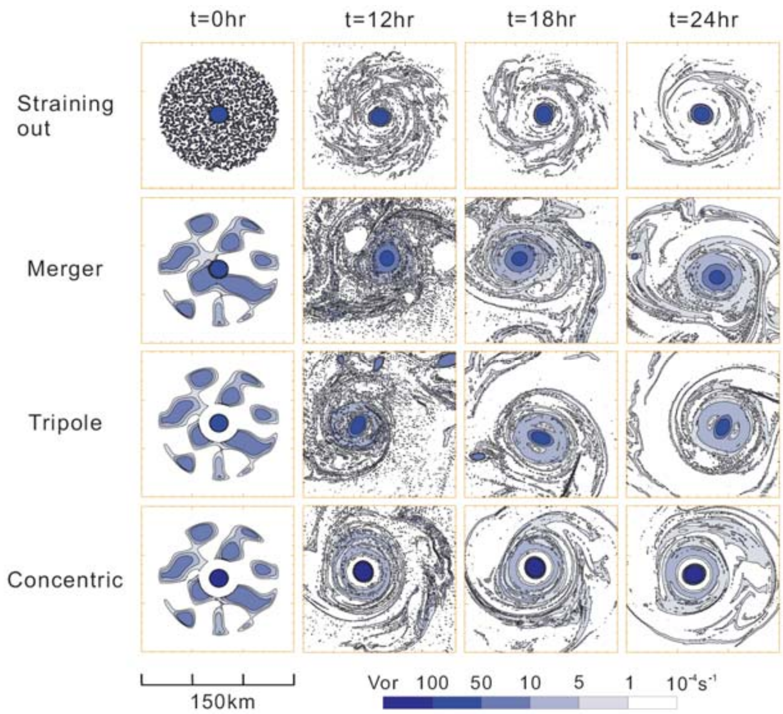
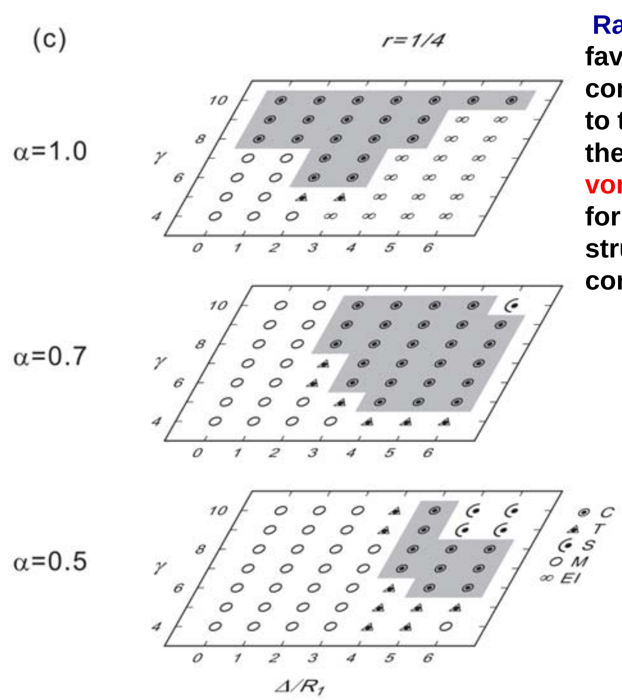


Figure 6: Similar to Figure 5 except that the dimensionless gap $\Delta/R_1=0$ and the vorticity strength ratio $\gamma=10$.

Examples of the vorticity field at hour 12, showing different classifications of binary vortex interactions involving a skirted core vortex.



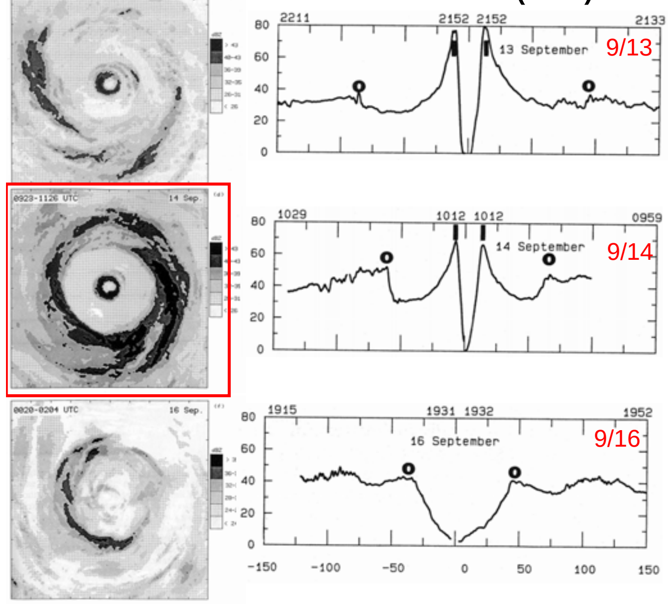




Rankine vortex (α = 1.0) favors the formation of a concentric structure **closer** to the core vortex, while the **α = 0.7 and α = 0.5 vortices** favor the formation of concentric structures **farther** from the core vortex.

A major issue in understanding changes in typhoon intensity

Black and Willoughby (1992)
Hurricane Gilbert (1988)

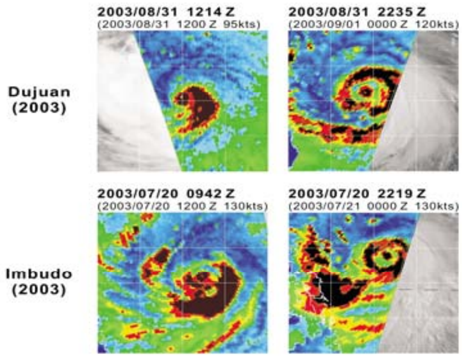


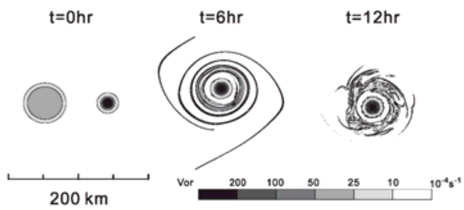
Development of symmetric structure from asymmetric convection in 12 hours

The contraction of the Outer tangential wind maximum

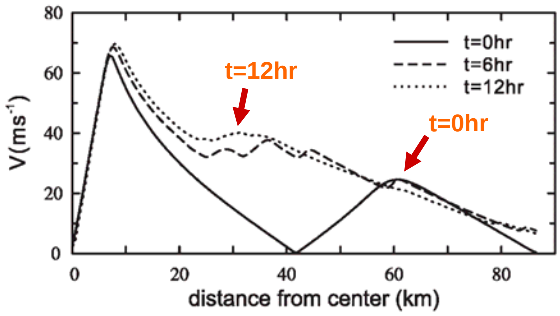
Core vortex intensity remains approximately the same during the contraction period

Inner core dissipate, TC weakens



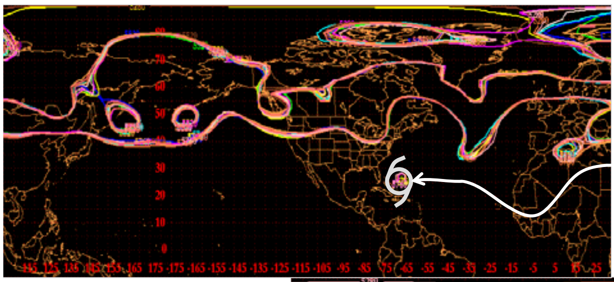


The contraction and the increase of the secondary wind maximum by nonlinear advection dynamics.



The Downstream Influences of the Extratropical Transition of Tropical Cyclones

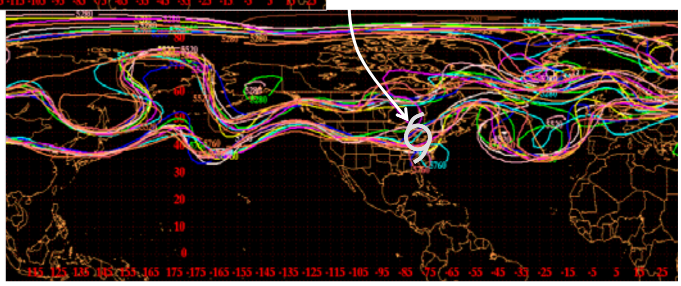
Patrick Harr
Naval Postgraduate School



0000 UTC 16 Sep 2003
GFS Ensembles +00

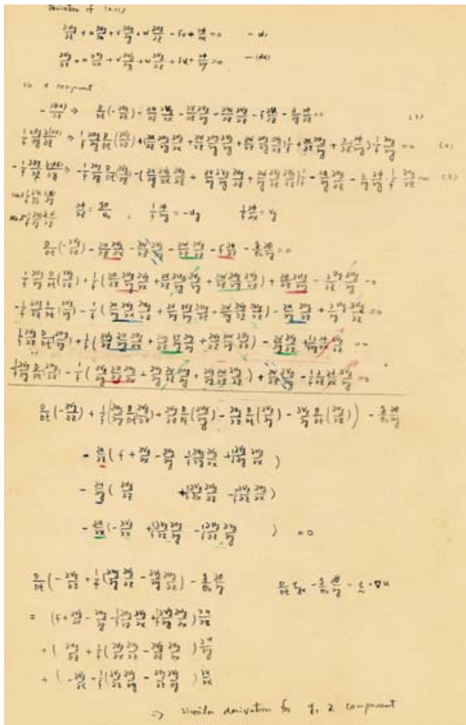
Hurricane Isabel

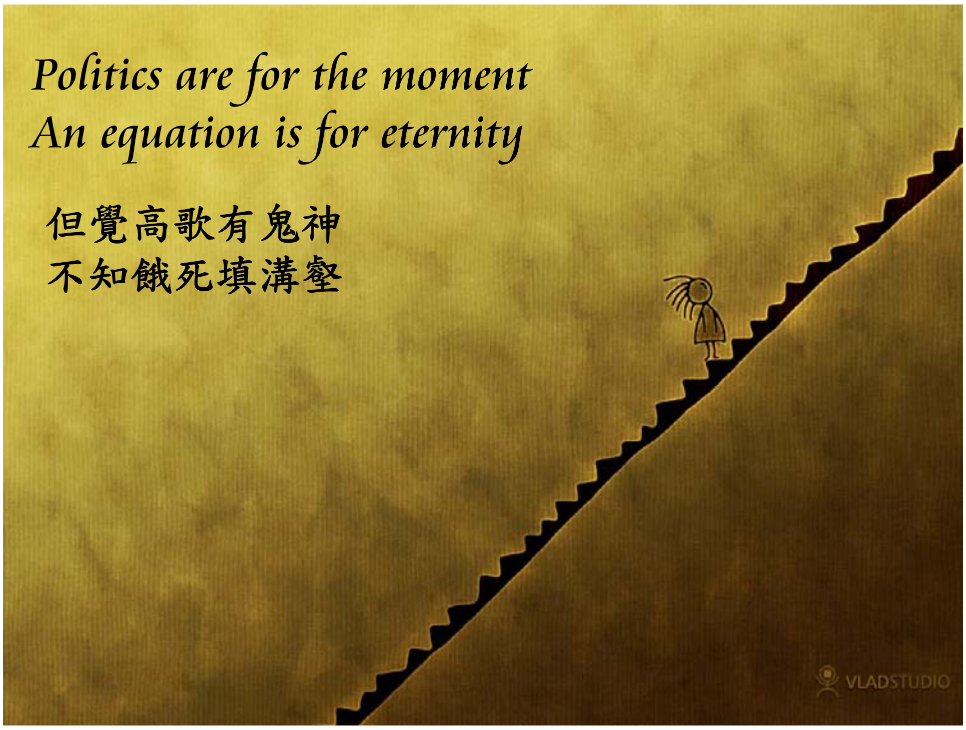
GFS 500 hPa Ensembles
+108 h
VT 1200 UTC 20 Sep 03



Acknowledgment: Office of Naval Research, Marine Meteorology Program

To derive the equations ..... or
not to derive the equations
that is a question !!





Same strain outside the RMW but different core vortex strength

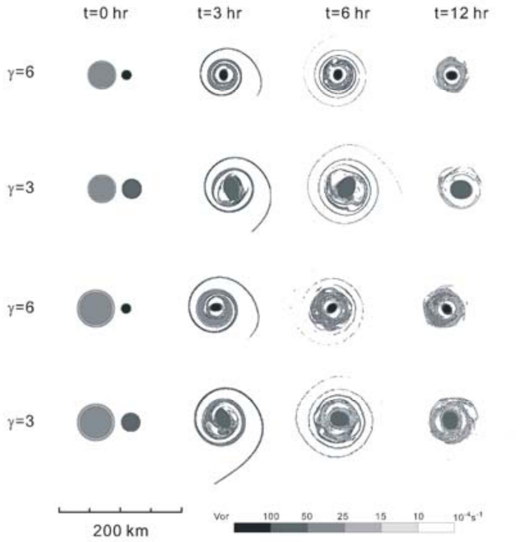
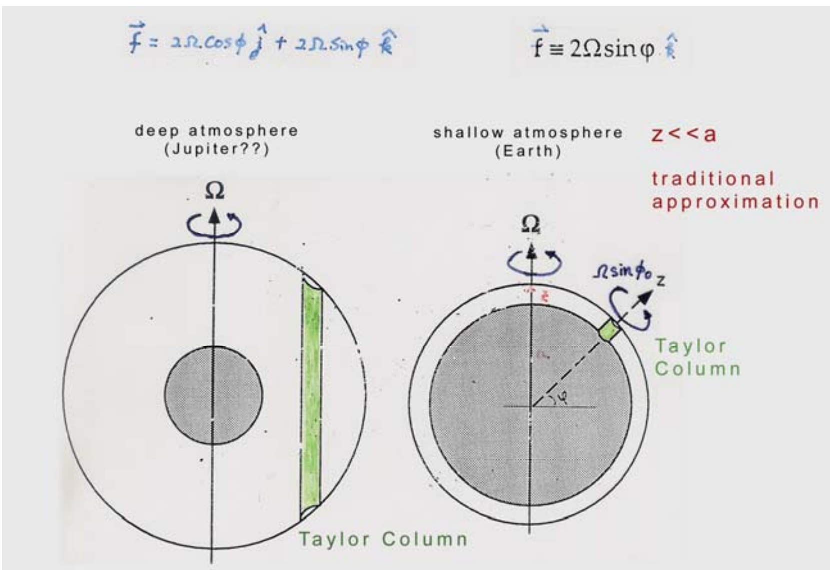


Figure 9: The sensitivity of the vorticity field in the binary vortex experiments with the core vortices process the same maximum wind but different radius of vorticity field. Two core vortices considered have the vorticity and radius of (1.8 × $10^{-2}s^{-1}$, 10 km) and (0.9 × $10^{-2}s^{-1}$, 20 km) respectively. The dimensionless gap is 1 in the experiments. The outer vortices considered have the radius of 30 km and 40 km respectively.



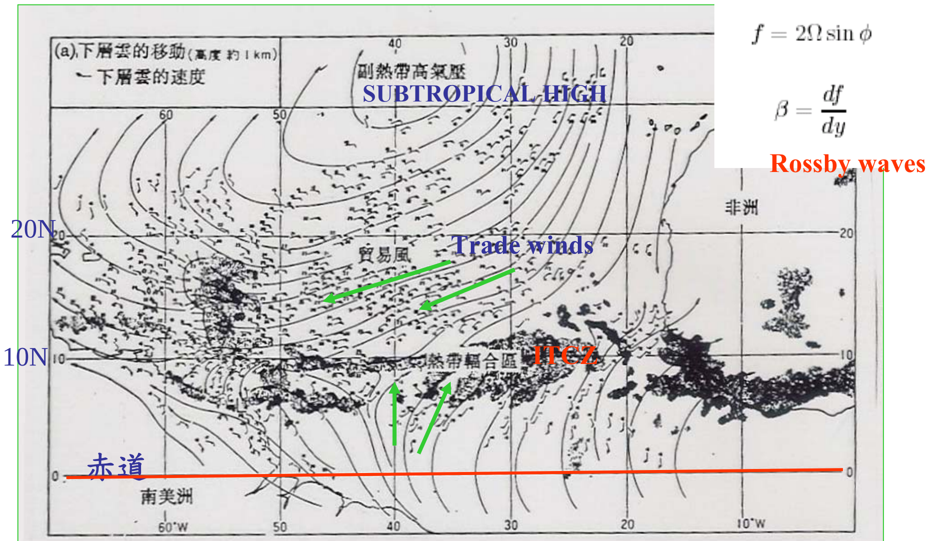
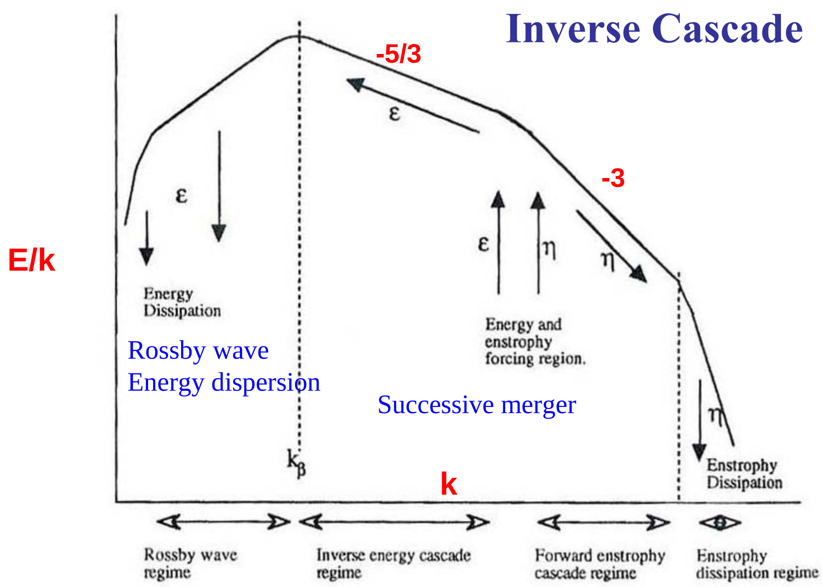


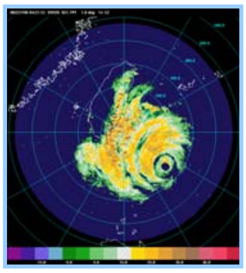
圖7.4 大西洋熱帶海洋區的風分佈圖（摘自T.Fujita,J.Meteor.Soc.Japan,49,1971），塗黑影區爲已發展之積雲系雲所存在的區域。圖爲國際標準時間1969年7月14日15時。

科氏力往北遞增

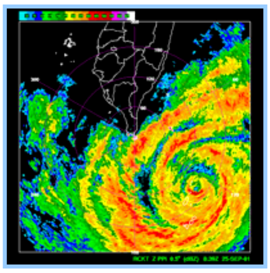
Waves, turbulence, and coherent vortex



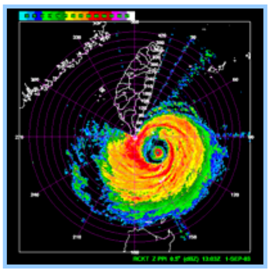
Concentric eyewalls near Taiwan



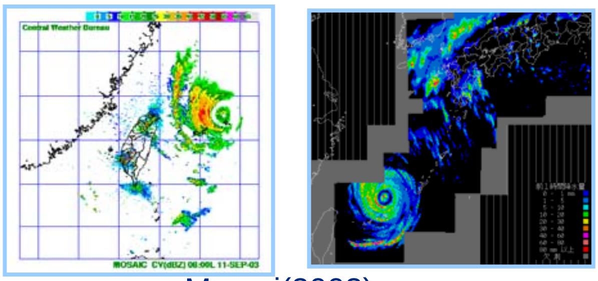
Bilis(2000)



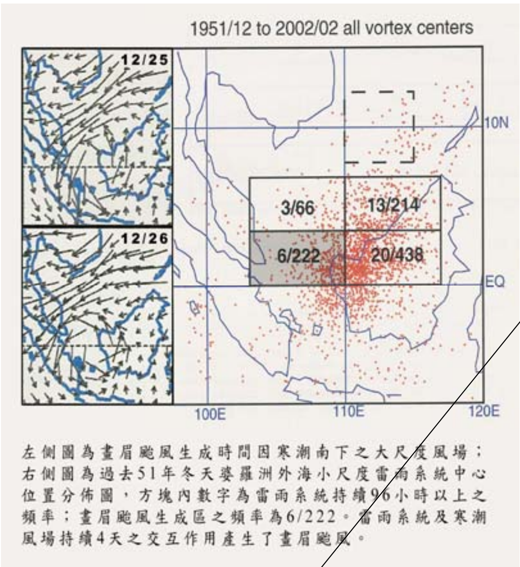
Lekima(2001)



Dujuan(2003)



Maemi(2003)

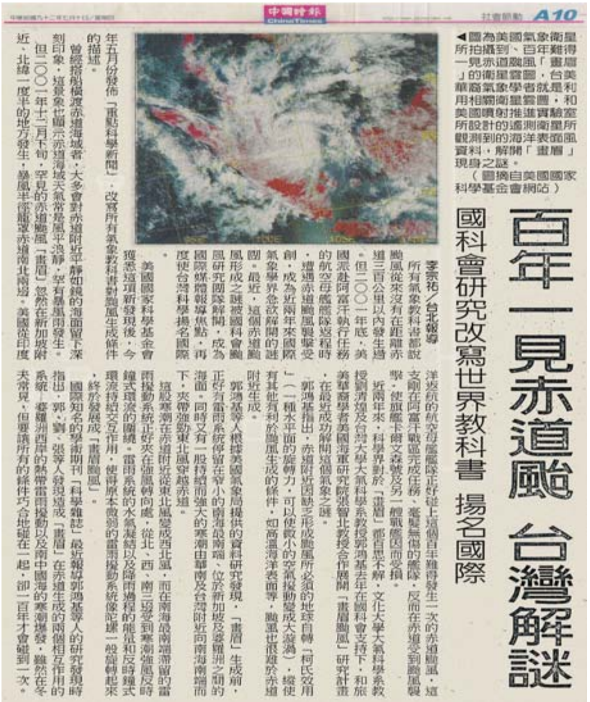


畫眉颱風

百年一見赤道颱風

寒潮Rossby wave train
風場環繞長生命期中
尺度雷雨系統

$$\frac{DP}{Dt} = \boxed{P}\left(\frac{j\cdot(\nabla\times F)}{j\cdot\zeta} + \boxed{\frac{k\cdot\nabla\dot{\theta}_\rho}{k\cdot\nabla\theta_\rho}} + \frac{\nabla\cdot(\rho_r U)}{\rho} \right)$$



社會脈動 A10

◀圖為美國氣象衛星所拍攝到、百年難得一見赤道颱風「畫眉」的衛星雲圖，台美華裔氣象學者就是利用相關衛星雲圖，和美國噴射推進實驗室所設計的遙測衛星所觀測到的海洋表面風資料，解開「畫眉」現身之謎。
（圖摘自美國國家科學基金會網站）

百年一見赤道颱 台灣解謎

國科會研究改寫世界教科書 揚名國際

Taylor's Experiment 1916

Strong Rotational Fluids

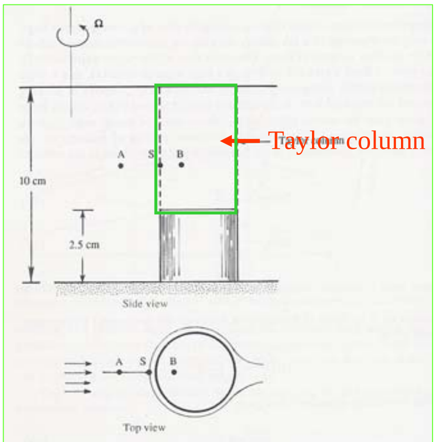


Fig. 13.6 G. I. Taylor's experiment in a strongly rotating flow of a homogeneous fluid.

Dye released from point A divided at point S, as if it had been blocked by an upward extension of the cylinder.

Dye released from a point B within the Taylor column remained there and moved with the cylinder.

↓

The polar vortex traps the ClO rich air over Antarctica for many months. The polar vortex also prevents the lateral mixing of warm air into the pole region.

The phenomenon is analogous to the horizontal blocking caused by a solid body in a strong stratified system.

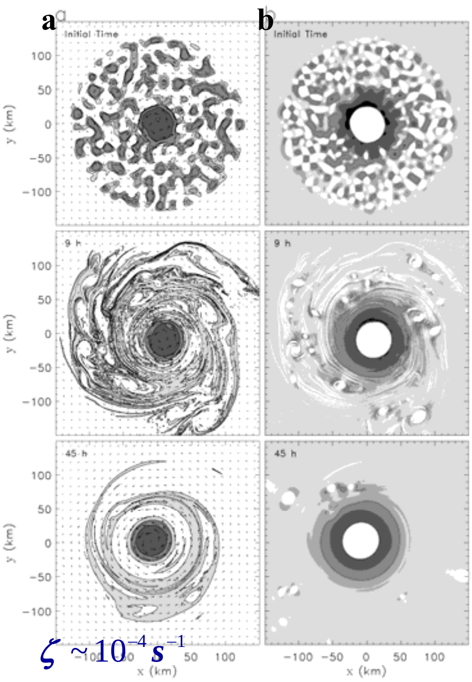
Burger Number Rossby number
Froude number

Shapiro and Willoughby (1982) and Schubert and Hack (1982) proposed that heating-vorticity interaction can lead to convective-ring contraction.

$$d\zeta/dt \sim \zeta \nabla \cdot \mathbf{V}$$

Stronger ζ near the TC core favors the inward response

Symmetrical Model



Rozoff et al. (2006)

The strong differential rotation outside the radius of maximum wind of the core vortex may also contribute to the formation and maintenance of the moat.

The rapid filamentation zones.

One way to produce a concentric vorticity structure is through the interaction between a strong core vortex and a background turbulent vorticity field.

a: vorticity field
b: filamentation time

Rozoff et al. (2006)

The strong differential rotation outside the radius of maximum wind of the core vortex may also contribute to the formation and maintenance of the moat.

The Rapid Filamentation Zone: A zone with the filamentation time smaller than the 30 min convective turnover time.

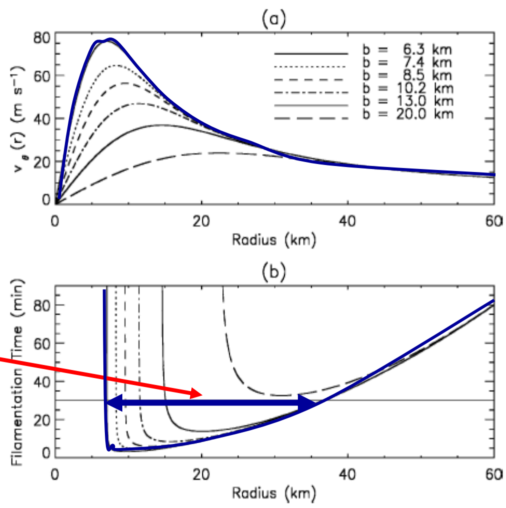


FIG. 2. (a) Radial profiles of v_θ (r) as given by (15) for six different values of the parameter b. (b) Corresponding radial profiles of $\tau_{fil}(r)$ as given by (16). The filamentation times $\tau_{fil}(r)$ are plotted only in the strain-dominated regions, where $S_1^2 + S_2^2 - \zeta^2 > 0$.

Summary

- **The results support the notion that concentric eyewalls form only in strong tropical cyclones.**
- **A strong tropical cyclone with a moat of 10-20 km width is able to organize a stirred vorticity field with 40-50 km spatial scale into a concentric structure similar to those formed in binary votex interactions.**
- **Rankine vortex favors the formation of a concentric structure closer to the core vortex, while the skirted vortices favor the formation of concentric structures farther from the core vortex.**
- **The pivotal role of the core vorticity strength in maintaining itself, in stretching, organizing and stabilizing the outer vorticity field. Furthermore, the results depict the shielding effect of the moat in preventing further merger and enstrophy cascade processes during concentric eyewall formation.**

Highly Nonlinear System

Turbulence, Order and Chaos
Multiple scale interaction
Cascade of kinetic energy and enstrophy
Deterministic and statistical dynamics

Laminar yields turbulence

Order (i.e. turbulence) emerges from chaos

Coherent structures emerge from chaos, under the action of an external constraint

Dimensional Analysis

2D Enstrophy cascade

$$\frac{\zeta^2}{l/u} = \frac{u^2/l^2}{l/u} = \frac{u^3}{l^3} = \chi$$

$$u = \chi^{\frac{1}{3}} l$$

$$u^2 l = u^2 k^{-1} = \chi^{\frac{2}{3}} l^3 = \chi^{\frac{2}{3}} k^{-3}$$

-3 spectrum

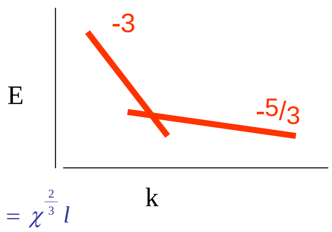
3D KE cascade

$$\frac{u^2}{l/u} = \varepsilon$$

$$u = \varepsilon^{\frac{1}{3}} l^{\frac{1}{3}}$$

$$u^2 l = u^2 k^{-1} = \varepsilon^{\frac{2}{3}} l^{\frac{5}{3}} = \varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}}$$

-5/3 spectrum



Energy Rich Meso-scale
Rich scale interactions

$$a = \frac{\partial KE}{\partial \chi} = \frac{u^2}{l} = \chi^{\frac{2}{3}} l$$

Less energetic sub scales

$$a = \frac{\partial KE}{\partial \chi} = \frac{u^2}{l} = \varepsilon^{\frac{2}{3}} l^{-\frac{1}{3}}$$

Very energetic sub scales

Multiple Scale Interactions

sub-scale turn over time

2-D

$$\tau \approx \frac{l}{u} = \chi^{-\frac{1}{3}}$$

Sub scale turn over time is the same

(a) vorticity gradient continues to increase until they are so large that diffusion can take over

(b) eddy turn over time constant with scale (worth the effort to Increase the model resolution)

(c) ζ constant with scale

(d) a decrease with scale

3-D

$$\tau \approx \frac{l}{u} = \varepsilon^{-\frac{1}{3}} l^{\frac{2}{3}}$$

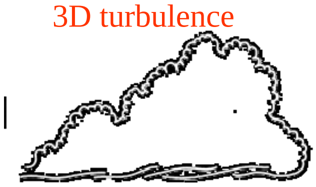
Fast turn over time
Full of surprises!

(b) eddy turn over time decrease with scale (very limited predictability gained when model resolution increased)

(c) ζ amplifies with decrease of scale

(d) a increase with decrease of scale

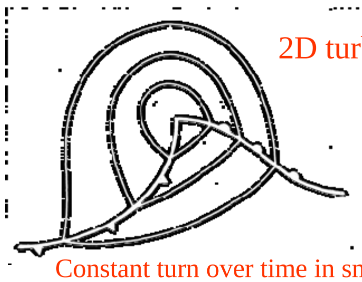
3D turbulence



Faster turn over time in small scale

Fig.1: A cumulus cloud is recognized most easily by its vigorous microstructure.

2D turbulence



Constant turn over time in small scale

Fig.2: A mid-latitude storm, shown here in a fairly early stage of its development, has smooth isobar patterns.

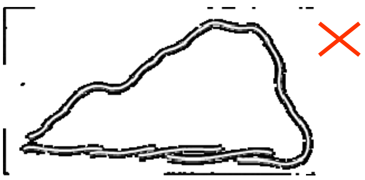


Fig.3: As successive smoothing operations are performed on the outline of the cloud in Fig.1, it becomes harder to recognize it as an actively entraining cumulus.

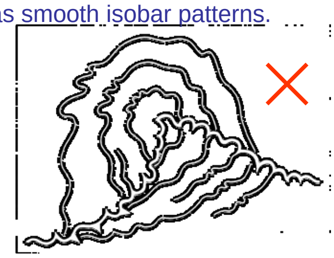
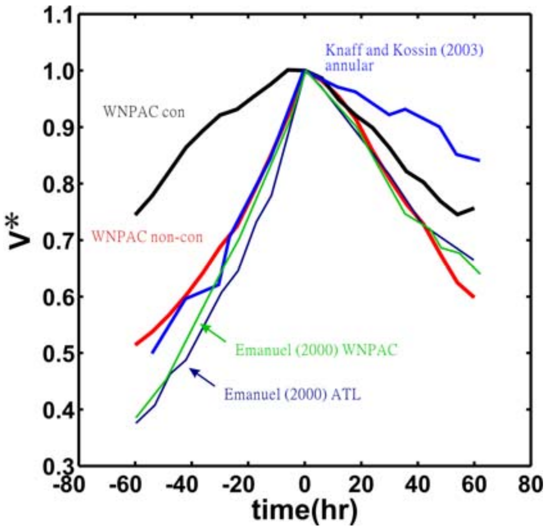
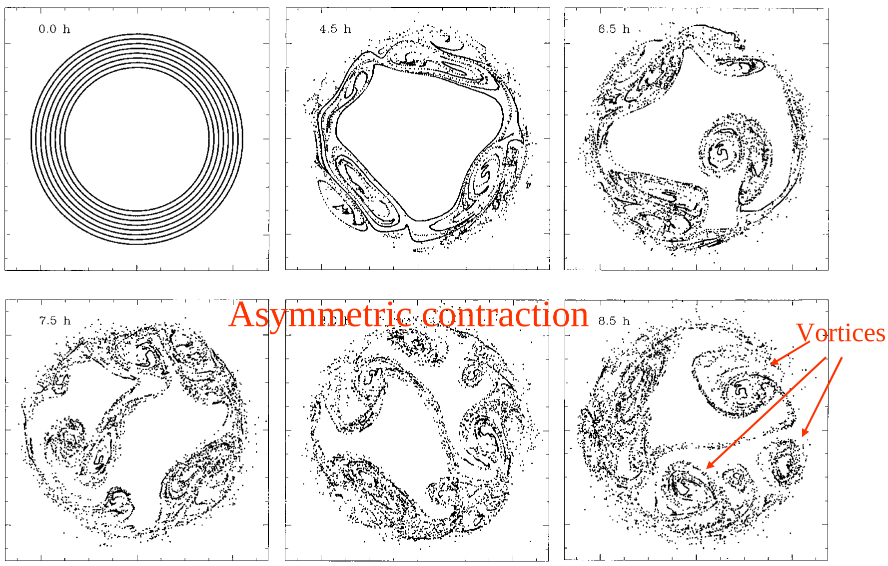


Fig.4: As storm of Fig.2 might look like this of it had microstructure dynamics similar to that of three-dimensional turbulence.

Kuo et al. 2007



Kossin and Schubert (2002)
Mixing due to Barotropic Instability



Advective rearrangement is different from the down-gradient diffusion