Exponential Function and Physical Examples

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Applied Math
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PART I: A derivation of exponential function

The discrete compound interest formula is

$$A = P(1 + \frac{r}{m})^{mt}$$

where A is the amount of money in the account,
P is the amount put into the account,
r is the annual percentage rate,
t is the number of years,
m the number of compoundings or payments per year.
The continuous compound formula is

$$A = Pe^{rt}$$

[from $dA/dt = rA$ and $A(0) = P$.]
By letting $x = rt$ and $n = mt$ we derive the celebrated definition of exponential function

$$e^x = \lim_{n\to\infty}(1 + \frac{x}{n})^n. \qquad (A1)$$

The binomial formula for positive integer is [**WHY?**]

$$(x + y)^n = \sum_{m=0}^{n} \binom{n}{m} x^{n-m} y^m, \qquad (A2)$$

where

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}.$$

Let $x = 1$ and $y = x/n$ in equation (A2), we yield

$$(1 + \frac{x}{n})^n = 1 + \frac{nx}{n} + \frac{n(n-1)}{2!}\frac{x^2}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{x^3}{n^3} + \cdots,$$

$$= 1 + x + \frac{1}{2!}(\frac{n}{n})(1 - \frac{1}{n})x^2 + \frac{1}{3!}(\frac{n}{n})(1 - \frac{1}{n})(1 - \frac{2}{n})x^3 + \cdots.$$

We then yield a very important expansion of exponential function when n is approaching infinite,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$

or

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \qquad (A3)$$

Equation (A2) is one of the most important formula in mathematical analysis. Note that the demarkation of the linear and nonlinear regimes in e^x is 1. By expanding e^{-x} based on (A3), we have

$$e^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$$

$$= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots,$$

which is also of enormous importance in the analysis.[Why there is the e^{-1} scale? Why not the e^{-2} or e^{-3} scale?]
Another tremendous important exponential function related formula is

$$e^{ix} = \cos x + i \sin x. \qquad (A4)$$

Equations (A3) and (A4) lead to two important formula for $\cos x$ and $\sin x$.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots,$$

and

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots.$$

Problems

(1) Derive $\cos(x+y) = \cos x \cos y - \sin x \sin y$ and $\sin(x+y) = \cos x \sin y + \sin x \cos y$ formula from $e^{i(x+y)}$ by using (A4) and equating the real and the imaginary part.

(2) Derive the differentiation of $\cos x$ and $\sin x$ by using (A4) and the formula $de^{ix}/dx = ie^{ix}$.

(3) Show that

$$\cos x = \frac{e^{ix} + e^{-ix}}{2},$$

and

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

(2) Consider the following differential equations which possess the exponential function related solutions ($\alpha > 0$):

$$\frac{dP}{dt} = \alpha P, \tag{1}$$

$$\frac{dP}{dt} = -\alpha P, \tag{2}$$

$$\frac{dP}{dt} = 1 - P. \tag{3}$$

Sketch diagrams for the solutions of (1), (2), and (3). Match the above equations with one of the following descriptions or equations. [Note that *A single differential equation can serve as mathematical model for many different phenomena.*]

(a) The rate of population growth at any given time is proportional to the population at that time. [One of the earliest attempts to model human population growth by means of mathematics was by the English economist Thomas Malthus in 1798. Basically, the idea behind the Malthusian model is the assumption that the rate at which the population of a country grows at a certain time is proportional to the total population of the country at that time.]

(b) Rate of a nuclei substance decay is proportional to the number of nuclei (radio-decay problem).

(c)The pollution level in a swimming pool if no new pollutants are added, only fresh water is introduced (and mixes immediately) and the rate of inflow and outflow are same. [see also the problem (l).]

(d) The amount of ultraviolet radiation being absorbed by O_3 is proportional to the intensity of the ultraviolet radiation. [Why does the maximum UV radiation absorption by O_3 occur in a altitude higher than the altitude of maximum O_3 concentration?]

(e) Percentage of the task learned at any time if the rate of learning is the percentage of a task **not** learned.

(f) The rate at which material is *forgotten* is proportional to the amount memorized in time t.

(g) The rate at which a subject is memorized is assumed to be proportional to the amount that is left to be memorized. Suppose M denotes the total amount of a subject to be memorized and $A(t)$ is the amount memorized in time t. The governing equation can be written as

$$\frac{dA}{dt} = \alpha(M - A).$$

(h) Newton's law of cooling/warming: the rate at which the temperature (T) of a body changes is proportional to the difference between the temperature of the body and the temperature of the surrounding media (T_m). What is the equilibrium temperature?

$$\frac{dT}{dt} = -\alpha(T - T_m)$$

(i) Falling body and air resistance problem maybe governed by the following equation

$$m\frac{dv}{dt} = mg - \alpha v.$$

What is the terminal velocity?

(j) The heating of a stove maybe governed by the following equation

$$c\frac{dT}{dt} = R - \alpha(T - T_m)$$

where c is the specific heat (比熱), R constant heat source, and α the heat transfer constant.

(k) Can you convert the equations in (g),(i) and (j) to the prototype form of (3)?

(l) Mixture of two salt solution of different concentration (and mixes immediately). Let A(t) [mol] denote the mole number of salt substance, V_0 $[m^3]$ the volume of the container, γ $[m^3/s]$ the input rate of solution with C_0 $[mol/m^3]$ concentration. The current concentration (which changes with time due to the mixture of C_0 solution) of the brine is A/V_0 $[mol/m^3]$. Why is the governing equation

$$\frac{dA}{dt} = \gamma C_0 - \gamma \frac{A}{V_0}.$$

(m) Continuous compound interest earned from a bank (interest earned is proportional to the money deposited in the bank).