SUMMARY OF CHAPTER 6
LINEAR ALGEBRA: MATRICES, VECTORS, DETERMINANTS
LINEAR SYSTEMS OF EQUATIONS

An $m \times n$ **matrix** $\mathbf{A} = [a_{jk}]$ is a rectangular array of numbers ("entries" or "elements") arranged in m horizontal rows and n vertical columns. If $m = n$, the matrix is called **square.** A $1 \times n$ matrix is called a **row vector** and an $m \times 1$ matrix a **column vector** (Sec. 6.1).

The **sum** $\mathbf{A} + \mathbf{B}$ of matrices of the same **size** (i.e., both $m \times n$) is obtained by adding corresponding entries. The **product** of $\mathbf{A}$ by a scalar c is obtained by multiplying each a_{jk} by c (Sec. 6.1).

The **product** $\mathbf{C} = \mathbf{AB}$ of an $m \times n$ matrix $\mathbf{A}$ by an $r \times p$ matrix $\mathbf{B} = [b_{jk}]$ is defined only when $r = n$, and is the $m \times p$ matrix $\mathbf{C} = [c_{jk}]$ with entries

(1) $$c_{jk} = a_{j1}b_{1k} + a_{j2}b_{2k} + \cdots + a_{jn}b_{nk}$$ (row j of $\mathbf{A}$ times column k of $\mathbf{B}$).

This multiplication is motivated by the composition of **linear transformations** (Secs. 6.2, 6.8). It is associative, but is ***not commutative:*** if **AB** is defined, **BA** may not be defined, but even if **BA** is defined, $\mathbf{AB} \neq \mathbf{BA}$ in general. Also $\mathbf{AB} = \mathbf{0}$ may not imply $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$ or $\mathbf{BA} = \mathbf{0}$ (Secs. 6.2, 6.7). Illustrations:

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

$$[1 \quad 2] \begin{bmatrix} 3 \\ 4 \end{bmatrix} = [11], \qquad \begin{bmatrix} 3 \\ 4 \end{bmatrix} [1 \quad 2] = \begin{bmatrix} 3 & 6 \\ 4 & 8 \end{bmatrix}.$$

The *transpose* $\mathbf{A}^{\mathsf{T}}$ of a matrix $\mathbf{A} = [a_{jk}]$ is $\mathbf{A}^{\mathsf{T}} = [a_{kj}]$; rows become columns and conversely (Sec. 6.1). Here, **A** need not be square. If it is and $\mathbf{A} = \mathbf{A}^{\mathsf{T}}$, then **A** is called **symmetric;** if $\mathbf{A} = -\mathbf{A}^{\mathsf{T}}$, it is called **skew-symmetric.** For a product, $(\mathbf{AB})^{\mathsf{T}} = \mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}$ (Sec. 6.2).

A main application of matrices concerns **linear systems of equations**

$$\mathbf{Ax} = \mathbf{b} \qquad \text{(Sec. 6.3)} \tag{2}$$

(m equations in n unknowns $x_1, \cdots, x_n$; **b** given). The most important method of solution is the **Gauss elimination** (Sec. 6.3), which reduces the system to "triangular" form by *elementary row operations,* which leave the set of solutions unchanged. (Numerical aspects and variants, such as *Doolittle's* and *Cholesky's methods,* are discussed in Secs. 18.1 and 18.2.)

Cramer's rule (Sec. 6.6) represents the unknowns in a system (2) of n equations in n unknowns as quotients of determinants; for numerical work it is impractical. **Determinants** (Sec. 6.6) have decreased in importance, but will retain their place in eigenvalue problems, elementary geometry, etc.

The **inverse** $\mathbf{A}^{-1}$ of a square matrix satisfies $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$. It exists if and only if $\det \mathbf{A} \neq 0$. It can be computed by the *Gauss–Jordan elimination* (Sec. 6.7).

The **rank** r of a matrix **A** is the maximum number of linearly independent rows or columns of **A** or, equivalently, the number of rows of the largest square submatrix of **A** with nonzero determinant (Secs. 6.4, 6.6).

The system (2) has solutions if and only if rank $\mathbf{A}$ = rank $[\mathbf{A} \quad \mathbf{b}]$, where $[\mathbf{A} \quad \mathbf{b}]$ is the *augmented matrix* (Fundamental Theorem, Sec. 6.5).

The *homogeneous system*

$$\mathbf{Ax} = \mathbf{0} \tag{3}$$

has solutions $\mathbf{x} \neq \mathbf{0}$ ("nontrivial solutions") if and only if rank $\mathbf{A} < n$, in the case $m = n$ equivalently if and only if $\det \mathbf{A} = 0$ (Secs. 6.5, 6.6).

Vector spaces, inner product spaces, and linear transformations are discussed in Sec. 6.8. See also Sec. 6.4.

SUMMARY OF CHAPTER 7
LINEAR ALGEBRA: MATRIX EIGENVALUE PROBLEMS

The practical importance of these problems cannot be overrated. They are defined by the equation

$$\mathbf{Ax} = \lambda\mathbf{x}. \tag{1}$$

A is a given square matrix. All matrices in this chapter are ***square.*** λ is a scalar. To *solve* the problem (1) means to determine values of λ, called **eigenvalues** (or **characteristic values**) of **A,** such that (1) has a nontrivial solution **x,** called an **eigenvector** of **A** corresponding to that λ. An $n \times n$ matrix has at least one and at most n numerically different eigenvalues. These are the solutions of the **characteristic equation** (Sec. 7.1)

$$D(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0. \tag{2}$$

$D(\lambda)$ is called the **characteristic determinant** of **A.** By expanding it we get the **characteristic polynomial** of **A,** which is of degree n in λ. Some typical applications are shown in Sec. 7.2.

Section 7.3 is concerned with eigenvalue problems for **symmetric** ($\mathbf{A}^\mathsf{T} = \mathbf{A}$), **skew-symmetric** ($\mathbf{A}^\mathsf{T} = -\mathbf{A}$), and **orthogonal matrices** ($\mathbf{A}^\mathsf{T} = \mathbf{A}^{-1}$). Section 7.4 presents eigenvalue problems for the complex analogs of these matrices, called **Hermitian** ($\bar{\mathbf{A}}^\mathsf{T} = \mathbf{A}$), **skew-Hermitian** ($\bar{\mathbf{A}}^\mathsf{T} = -\mathbf{A}$), and **unitary matrices** ($\bar{\mathbf{A}}^\mathsf{T} = \mathbf{A}^{-1}$). All the eigenvalues of a Hermitian matrix (and a symmetric one) are real. For a skew-Hermitian (and a skew-symmetric) matrix they are pure imaginary or zero. For a unitary (and an orthogonal) matrix they have absolute value 1.

The diagonalization of matrices and the transformation of quadratic forms to principal axes are related to eigenvalues, as explained in Sec. 7.5.