5th Homework
for Lifescience Mathematics and Applied Mathematics

Matrices, Eigenvalues, Eigenvectors, and Others

(1) Consider the differential equation

$$\frac{d^2y}{dt^2} = -k^2 y. \tag{1}$$

(a) show that $\cos kt$, $\sin kt$, e^{ikt}, and e^{-ikt} are solutions of (1).

(b) let $dy/dt = x$ and define the column vector $\mathbf{u} = (x, y)^T$, we can write the equation (1) in matrix form,

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u}.$$

What is matrix A and what is the eigenvalues λ_1, λ_2 of A.

(c) show that the $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ are solutions of (1).

(d) compare and discuss your (c) and (a) solutions.

(2) Consider the differential equation

$$\frac{d^2y}{dt^2} = k^2 y. \tag{2}$$

(a) show that e^{kt}, and e^{-kt} are solutions of (2).

(b) let $dy/dt = x$ and define the column vector $\mathbf{u} = (x, y)^T$, we can write the equation (2) in matrix form,

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u}.$$

What is matrix A and what is the eigenvalues λ_1, λ_2 of A.

(c) show that the $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ are solutions of (2).

(d) compare and discuss your (c) and (a) solutions.

(3) The so-called inertial oscillation in the ocean and in the atmosphere or the Foucaut Pendulum problem are governed by the following equations

$$\frac{du}{dt} = fv, \tag{3}$$

$$\frac{dv}{dt} = -fu, \tag{3}$$

where u, v are east-west and north-south velocity components and $f = 2\Omega \sin\phi$ is the Coriolis parameter.

(a) write the (3) in the matrix form.
(b) find the eigenvalues and eigenvectors of the matrix.
(c) discuss the phase and periodicity of the solutions.

(4) For a second order ordinary differential equation take the form of (i.e. $a \neq 0$)

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0, \tag{4}$$

we can get the so-called auxiliary equation

$$ar^2 + br + c = 0. \tag{5}$$

(a) show that (5) can be resulted from plugging e^{rt} into (4).
(b) by algebra we know that (5) has two roots

$$r_1 = \frac{-b + \sqrt{Q}}{2a},$$

and

$$r_2 = \frac{-b - \sqrt{Q}}{2a},$$

where $Q = b^2 - 4ac$.
(c) if r_1 and r_2 both are real and are unequal ($Q > 0$), show that

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

is the general solution of (4).
(d) if r_1 and r_2 both are real and are equal ($Q = 0$, $r_1 = r_2 = r$), show that

$$y = c_1 e^{rt} + c_2 t e^{rt}$$

is the general solution of (4).
(e) if r_1 and r_2 both are complex ($Q < 0$), and that $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$ show that

$$y = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t)$$

is the general solution of (4).
(f) write (4) in matrix form and show that the characteristic polynomial of the matrix is the same as (5).
(g) what are the eigenvalues of the matrix?

(5) A bottle of soda pop is at room temperature 30°C is placed in a refrigerator where the temperature is 6°C. After half an hour the soda pop has cooled to 10°C.

(a) what is the temperature of the soda pop after another half hour?

(b) compare the cooling rate of the soda in the first and second half hour.

(c) How long does it take the soda pop to cool to 6°C (in equilibrium with the refrigerator temperature.)

(d) If you are a police detective or a forensic laboratory assistant, can you describe how will you quickly estimate the time of death for a murdered victim just gunned down a short while ago?

(6) Solve the following ODE with y_0 as initial or boundary condition.

(a)

$$\frac{dy}{dx} + y = 2,$$

(b)

$$\frac{dy}{dx} + \frac{y}{x} = e^x,$$

(c)

$$\frac{dy}{dx} + xy = 2,$$

(d)

$$\frac{dy}{dx} = y^2 e^{-x},$$

(e)

$$x\frac{dy}{dx} = \frac{y^2}{x} + y,$$

(f)

$$\frac{dy}{dt} = ky(1 - \frac{y}{m}),$$

(g)

$$C\frac{dy}{dt} = R - \alpha(y - y_0),$$

(h)

$$(x - 2)\frac{dy}{dt} = x - y,$$

(i)

$$\frac{dy}{dt} = \lambda y + f_0 e^{i\Omega t},$$

(j)

$$\frac{dy}{dt} = -\alpha(y - m).$$

Bonus Problem

(7) Evaluate the following equations ($n = 0, 1, ...,$ which are all positive integers)

(a)

$$\sum_{n=0}^{\infty}(nh\nu)e^{-\frac{nh\nu}{KT}} =?$$

(b)

$$\sum_{n=0}^{\infty}e^{-\frac{nh\nu}{KT}} =?$$

(c)

$$\int_0^{\infty} Ee^{-\frac{E}{KT}}dE =?$$

(d)

$$\int_0^{\infty} e^{-\frac{E}{KT}}dE =?$$

Here h ($6.6262 \times 10^{-34} Js$), K ($1.381 \times 10^{-23} Jdeg^{-1}$) are the **Planck and Boltzman constants**, ν frequency, and T absolute temperature. These equations are first derived in the end of the 19 century for the celebrated Planck and Rayleigh-Jean law for the spectral energy density of the black body radiation. These derivations mark the beginning of the quantum mechanics.

(e) what are the physical meanings of

$$e^{-\frac{nh\nu}{KT}},$$

and

$$e^{-\frac{E}{KT}}.$$