4th Homework and Notes for Lifescience Mathematics

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10/26/2005

One of the most interesting contexts in which nonlinear system of differential equations arise is the modeling of interacting populations. Such *ecological* models first appeared in the independent work of Lotka and Volterra in the 1920s and comprise a cornerstone of *mathematical biology*, or *biomathematics*. The following nonlinear systems are some examples from the biology.

I Malthusian and Logistic Models The Multhusian model, so named after the English economist Thomas Malthus of the late eighteenth century. The Malthusian model, $dP/dt = kP$, is not realistic for a population that is naturally limited by environmental factors such as finite space and finite food supply. A simple model that takes into account the sustainbility of the environment to the population growth is the *logistic* model. The logistic model is credited to Pierre Verhulst, a Belgian mathematician of the mid-nineteenth century. The model is

$$\frac{dP}{dt} = kP(1 - \frac{P}{M}), \quad P(0) = P_0.$$

The k is called the intrinsic growth rate and M is the **carrying capacity** of the environment. Explain why ? Also, can you find the solution of the equation to be

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-kt}}.$$

The logistic model can include the effect of **harvesting** on such a population. By harvesting, we continuous, deliberate removal of members of the population at some specified rate. Two types of harvesting are often considered, *proportional harvesting* and *constant harvesting*. Namely, addition of $-\beta P$ term ($-R$term) in the logistic equation for the *proportional harvesting*(*constant harvesting*)effect.

II Model of Competition

Two species compete for resources that each requires in order to live, such as food or territory. It is interesting to note that, through the analysis of model, the general conclusion is *whether it is weak or strong, a species survives in the long run if its competitor is weak. If both species are strong, then the surviving species is determined by initial conditions.* A species is *weak (strong)* if its competitive strength is less (greater) than unity in the following equation.

Equation set (**1**)

(i) The population growth rates, dx/dt, and dy/dt, decrease proportionally with increasing competitor population.

(ii) In the absence of its competitor, each of the species is governed by a simple logistic equation with some intrinsic growth rate and environmental carrying capacity.

$$\frac{dx}{dt} = k_1 x(1 - x - ay)$$

$$\frac{dy}{dt} = k_2 y(1 - y - bx)$$

What is the intrinsic growth rate and environmental carrying capacity in the equations?

Explain the equations (the meaning each term in the equations).

III Model of Predation

Two species live within the same environment, one species the prey, is the food source for the other species, the predator. The prey's food source is an abundant third organism. It is interesting to note that, through the analysis of model, the general conclusion is *if the predator is efficient, as indicated by $\epsilon > 1$, then the predator population survives in the long run. Otherwise the predator drives itself toward extinction by consuming too much and reproducing too little.*

Equation set (**2**)

(i) In the absence of predators, the prey population is governed by a simple logistic equation with intrinsic growth rate and environmental carrying capacity.

(ii) In the absence of prey, per capita growth rate of the predator population is a negative constant (i.e. the predator population declines exponentially).

(iii) The per capita growth rate of the prey population, decreases proportionally with increasing predator population.

(iv) The per capita growth rate of the predator population, increases proportionally with increasing prey population.

$$\frac{dx}{dt} = k_1 x(1 - \frac{x}{\epsilon} - y)$$

$$\frac{dy}{dt} = -k_2 y(1 - x)$$

What are x and y (predator or prey)?

Explain the equation (the meaning of each term in the equations).

IV Modified Competition models

Equation set (**3**)

The carrying capacity for each species is inversely proportional to the square root of the population size of the other species.

$$\frac{dx}{dt} = x(1 - x\sqrt{y})$$

$$\frac{dy}{dt} = y(1 - y\sqrt{x})$$

What is the intrinsic growth rate and environmental carrying capacity in the equations?

Explain the equations (the meaning of each term in the equations).

Equation set (**4**)

The competitive strength of the x-population decrease as x increase.

$$\frac{dx}{dt} = x(1 - \frac{1}{2}x - y)$$

$$\frac{dy}{dt} = y(1 - y - \frac{x}{1 + x})$$

Explain the equations (the meaning of each term in the equations).

Equation set (**5**)

In the absence of competition, the per capita rate of decline of large populations is bounded.

$$\frac{dx}{dt} = k_1 x(\frac{1}{x} - 1 - ay)$$

$$\frac{dy}{dt} = k_2 y(\frac{1}{y} - 1 - bx)$$

Explain the equations (the meaning of each term in the equations).

V Modified Predator and Prey models

Equation set (**6**)

Predatory efficiency decreases with increasing numbers of predators.

$$\frac{dx}{dt} = x(1 - \frac{1}{2}x - y)$$

$$\frac{dy}{dt} = -\frac{1}{2}y(1 - x + y)$$

Explain the equations (the meaning of each term in the equations).

Equation set (**7**)

Predatory efficiency increases with y when y is small but eventually decrease as y increases.

$$\frac{dx}{dt} = x(1 - x - \frac{1}{2}y)$$

$$\frac{dy}{dt} = -\frac{1}{10}y(1 - x + y(y - 2))$$

Explain the equations (the meaning of each term in the equations).

Equation set (**8**)

In the absence of predators, the per capita rate of decline of large prey population is bounded.

$$\frac{dx}{dt} = \frac{1}{2}x(-1 + \frac{2}{x} - y)$$

$$\frac{dy}{dt} = -y(1 - x)$$

Explain the equations (the meaning of each term in the equations).

Equation set (**9**)

The carrying capacity for the predator population is proportional to the number of prey.

$$\frac{dx}{dt} = \frac{1}{3}x(1 - x - y)$$

$$\frac{dy}{dt} = -\frac{1}{10}y(1 - \frac{y}{2x})$$

Explain the equations (the meaning of each term in the equations).

Equation set (**10**)

Predators migrate into the environment at a rate proportional to the number of prey.

$$\frac{dx}{dt} = \frac{1}{3}x(1 - \frac{1}{2}x - y)$$

$$\frac{dy}{dt} = -y(1 - \frac{1}{2}x) + \frac{1}{2}x$$

Explain the equations (the meaning of each term in the equations).

VI Models of two cooperative populations

Equation set (**11**)

$$\frac{dx}{dt} = k_1 x(1 - \frac{x}{\epsilon} + y)$$

$$\frac{dy}{dt} = k_2 y(1 - \frac{y}{\beta} + x)$$

Why is this a cooperative system?

VII Three-species Community Models

Equation set (**12**)

Two predators and one prey system with competition between the predator species.

$$\frac{dx}{dt} = x(1 - x - \frac{1}{2}y - z)$$

$$\frac{dy}{dt} = y(-1 + 2x - z)$$

$$\frac{dz}{dt} = z(-1 + 3x - 2y)$$

Describe the relationship among the x, y and z species.

Find the equilibrium points at which $x, y, z \geq 0$, and determine whether each is stable of unstable.

Equation set (**13**)

$$\frac{dx}{dt} = x(-1 - y + 2z)$$

$$\frac{dy}{dt} = y(-1 + 2x - 2z)$$

$$\frac{dz}{dt} = z(-1 - 3x + 2y)$$

Describe the relationship among the x, y and z species.

Find the equilibrium points at which $x, y, z \geq 0$, and determine whether each is stable of unstable.

Equation set (**14**)

$$\frac{dx}{dt} = x(1 - x + y)$$

$$\frac{dy}{dt} = y(1 - y + x - z)$$

$$\frac{dz}{dt} = z(-1 + 2y)$$

Describe the relationship among the x, y and z species.

Find the equilibrium points at which $x, y, z \geq 0$, and determine whether each is stable of unstable.

Equation set (**15**)

Consider a three-species community of lions, zebra, and grass. Lions feed upon the zebra, which in turn feed upon the grass.

$$\frac{dx}{dt} = x(1 - y)$$

$$\frac{dy}{dt} = y(-1 + 2x - z)$$

$$\frac{dz}{dt} = z(-1 + y)$$

What are x, y, and z species?

Equation set (**16**)

Consider a three-species community of pine trees, oak trees, and squirrels. Pine trees and oak trees compete for sunlight and nutrients, and squirrels feed upon acorns from the oak trees.

$$\frac{dx}{dt} = x(1 - x - 2y)$$

$$\frac{dy}{dt} = y(1 - x - y - z)$$

$$\frac{dz}{dt} = z(-1 + 3y)$$

What are x, y, and z species? Explain your solution.

———Notes on Matrices———

(1) Similarity transform of $\mathbf{A}$ is $\mathbf{S}^{-1}\mathbf{A}\mathbf{S}$, provided that $\mathbf{S}$ is nonsingular ($det\mathbf{S} \neq 0$).

(2) Let $\mathbf{P}$ and $\mathbf{D}$ be eigenvector and eigenvalue matrices of the matrix $\mathbf{A}$ respectively. The eigenvector matrix $\mathbf{P}$ has eigenvectors as its columns and the eigenvalue matrix is a diagonal matrix with all the eigenvalues are the diagonal components. The eigen-relationship is

$$\mathbf{AP} = \mathbf{PD},$$

and the diagonalization equation

$$\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}.$$

Note that the diagonalization is a special form of the similarity transform.

(3)
Symmetric matrix: $\mathbf{A}^T = \mathbf{A}(a_{ji} = a_{ij})$
Hermitian matrix: $\mathbf{A}^H = \mathbf{A}(a^*_{ji} = a_{ij})$
skew-symmetric matrix: $\mathbf{A}^T = -\mathbf{A}(a_{ji} = -a_{ij})$
skew-Hermitian matrix: $\mathbf{A}^H = -\mathbf{A}(a^*_{ji} = -a_{ij})$, where * is the complex conjugate value.
The eigenvalues of Hermitian (skew-Hermitian) matrix is real (imaginary). The eigenvectors from either Hermitian or skew-Hermitian form a complete vector space and eigenvectors are orthogonal. In another words, the normalized eigenvectors (vector norm or length is equal to unity) form an orthonormal set. [Sturm-Liouville theorem]

(4) if $\mathbf{A}$ is Hermitian or skew-Hermitian, then the $\mathbf{P}$ form from orthonormal eigenvectors as its columns possesses the property of

$$\mathbf{P}^H = \mathbf{P}^{-1}$$

(5) A system of dynamical equations of

$$\frac{d\mathbf{u}}{dt} = \mathbf{Au}$$

can always be written as

$$\mathbf{P}^{-1}\frac{d\mathbf{u}}{dt} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{P}^{-1}\mathbf{u},$$

$$\mathbf{P}^{-1}\frac{d\mathbf{u}}{dt} = \mathbf{D}\mathbf{P}^{-1}\mathbf{u}.$$

If $\mathbf{A}$ is Hermitian or skew-Hermitian, then the above the equation can be written (with the help of $\mathbf{P}^H = \mathbf{P}^{-1}$)

$$\mathbf{P}^H\frac{d\mathbf{u}}{dt} = \mathbf{D}\mathbf{P}^H\mathbf{u},$$

or

$$\frac{d\mathbf{v}}{dt} = \mathbf{D}\mathbf{v},$$

where $\mathbf{v}$ is the projection of $\mathbf{u}$ onto different eigenvectors. $\mathbf{v} = \mathbf{P}^H\mathbf{u}$ is the *projection* formula and the inverse relationship $\mathbf{u} = \mathbf{P}\mathbf{v}$ is the *expansion* formula. Note that $\mathbf{v}$ equations are uncoupled equations with each v_i component possesses $\exp(\lambda_i t)$ time behavior. The imaginary (real) eigenvalues λ_i corresponding oscillation (decay or growth) modes. The eigenvectors often called the **normal modes** of the system.

(6) We claim that

$$\frac{d\mathbf{u}}{dt} = \mathbf{A}\mathbf{u} + \mathbf{f},$$

is a generalized dynamical system with forcing.
We can study the above system via the coefficent of the single projected (normal) mode

$$\frac{dv}{dt} + (\alpha + i\omega)v = g.$$

An interesting example of the above equation is when the forcing is a periodic function, (i.e., $g = f_0 \exp(i\Omega t)$,

$$\frac{dv}{dt} + (\alpha + i\omega)v = f_0 \exp(i\Omega t).$$

The solution is

$$v(t) = (v(0) - Rf_0)\exp(-(\alpha + i\omega)t) + Rf_0 \exp(i\Omega t),$$

where $R = 1/(\alpha + i\omega + i\Omega)$, the response function.