Applied Mathematics

Note 3 and Homework 3

Ref: An introduction to ODE, by J. C. Robinson, and ZC

Radioactive Decay

Let N(t) denote the number of radioactive atoms in some sample material at time t. Then with $k > 0$ the equation

$$\frac{dN}{dt} = -kN, \tag{1}$$

is a very good model for the way that the number of radioactive atoms decays. The solution of (1) is

$$N(t) = N(t_0)e^{-k(t-t_0)}, \tag{2}$$

where $N(t_0)$ is the number of isotpes at time t_0.Equation (2) indicates that the number of radioactive isotopes decays exponentially to zero. The *half-life* of a particular radioactive isotope is the time it takes for half of the radioactive isotpes to decay, and this is related to the constant k that appears in equation. To find this relationship, suppose that there are N_0 radioactive atoms at time $t = 0$. Then the solution of (2) is

$$N(t) = N_0 e^{-kt}.$$

Half of the atoms will have decayed by time $t_{1/2}$ when $N(t_{half}) = 1/2N_0$, i.e.,

$$N_0 e^{-kt_{1/2}} = \frac{1}{2}N_0.$$

Taking the natural logarithm of both sides gives

$$-kt_{1/2} = -\ln 2,$$

and so the half-life is given by

$$t_{1/2} = \frac{\ln 2}{k}.$$

Note that this time does **not** depend on the initial number of radioactive atoms.

Radiocarbon Dating

About 1950 the chemist Willard Libby devise a method of using radioactive carbon as a means of determining the approximate ages of fossils. The theory of **carbon dating** is based on the fact that the isotope carbon-14 is produced in the atmosphere by the action of cosmic radiation on nitrogen. The ratio of the amount of C-14 to ordinary carbon in the atmosphere appears to be a constant, and as a consequence the proportion amount of the isotope present in all living organisms is the same as that in the atmosphere. The solution of (2) then forms the basis of the technology of radioactive dating. The essence of the method is as follows. Living matter is constantly taking up from the air. The result is that within such material the ratio of the number of isotopes of radioactive carbon 14 (^{14}C) to the number of isotopes of stable carbon 12 (^{12}C) is essentially constant. Once the specimen is dead (for example, a tree is cut down for its wood, or cotton is harvested for weaving), the radioactive ^{14}C atoms begin to decay according to the model (1). Since the half life of carbon 14 is approximately 5700 years, we need to take the constant k in (1) to be

$$k = \frac{\ln 2}{5700} \approx 1.216 \times 10^{-4}.$$

By examining the ratio of the number of isotopes of carbon 12 to carbon 14 in a sample of the material that we want to date, it is possible to work out the proportion remaining of the ^{14}C atoms that were initially present. Suppose that the sample stopped taking up carbon from the air when time $= t_0$, and that the number of ^{14}C atoms present then was $N(t_0)$. If we know that the sample now (at time t_1) contains only a fraction p of the initial level of ^{14}C, then $N(t_1) = pN(t_0)$.
Using our explicit solution $N(t) = N(t_0)e^{-k(t-t_0)}$, we should have

$$pN(t_0) = N(t_1) = N(t_0)e^{-k(t_1-t_0)}.$$

Cancelling the factor of $N(t_0)$ in the two outside terms yields the equation

$$p = e^{-k(t_1-t_0)}$$

Taking logarithms of both sides we have

$$\ln p = -k(t_1 - t_0),$$

and so the year t_0 from which the sample dates is given by

$$t_0 = t_1 + \frac{\ln p}{k}. \tag{3}$$

For his work Libby won the Nobel Prize for chemistry in 1960. Libby's method has been used to date wooden furniture in Egyptian tombs, the woven flax wrappings of the Dead Sea scrolls, and the cloth of the enigmatic shroud of Turin.

The Shroud of Turin, which shows the negative image of a body of a crucified man, is beleived by many to be the burial shroud of Jesus of Nazareth. See attached Fig. 3.9 for the picutre of the Shroud of Turin. In 1988 the Vatican granted permission to have shroud carbon dated.three independent groups of scientists from Arizona, Oxford and Zurich analyze the cloth. Fibers from the shroud were found to contain about 92% of the level in living matter. Using the expression in (3) shows that the Shroud therefore dates from

$$t_0 = 1988 + \frac{\ln 0.92}{0.0001216} \approx 1302,$$

putting its origin squarely in the Middle Ages.

Excercises

(1) Show that (2) is the solution of (1) with the $N(t_0)$ condition by plugging (2) into (1).

(2) In 1947 a large collection of papyrus scrolls, including the oldest known manuscript version of prtions of the Old Testament, was found in a cave near the Dead Sea: they have come to known as the "Dead Sea Scrolls". The scroll containing the book of Isaiah was dated in 1994 using the radiocarbon technique; it was found to contain between 75% and 77 % of the initial level of carbon 14. Between which dates was the scroll written?

(3) A large round table hangs on the wall of the castle in Winchester. Many would like to believe that this is the Round Table of King Arthur, who was at the height of his powers in about AD 500. If the table dates from this time, what proportion of the original carbon 14 would remain? In 1976 the table was dated using the radiocarbon technique, and 91.6% of the original quantity of carbon 14 was found. From when does the table date?

(4) Archaeologists used pieces of burned wood, or charcoal, found at the site to date prehistoric paintings and drawing on the walls and ceilings of a cave in Lascaux, France. See attached Fig. 3.8. Determine the approximate age of the paintings, if it was found that 85.5 % of the ^{14}C had decayed.

(5) Radiocarbon dating is an extremely delicate process. Discuss the accuracy of the method.

(6) The radioactive isotope of lead, Pb-209, decays at a rate proportional to the amount present at time t and has a half-life of 3.3 hours. If 1 gram of lead is present initially, how long will it take for 90 % of the lead to decay?

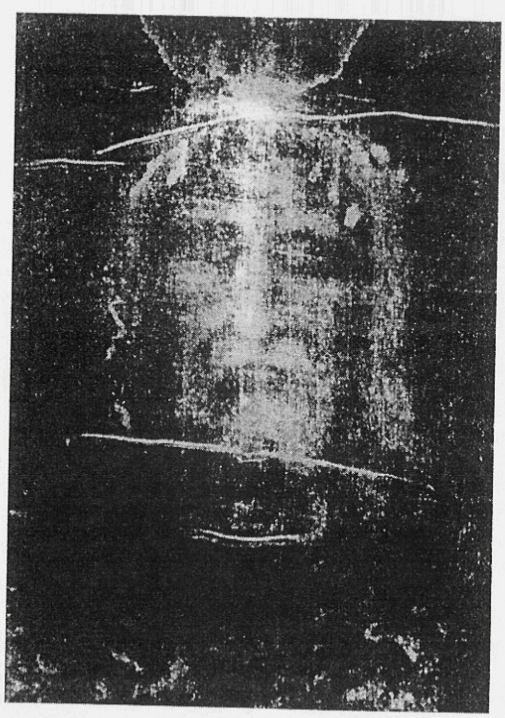

Figure 3.9 Shroud of Turin

Figure 3.8 Lascaux cave painting