Applied Mathematics 生命科學數學期中考 11/25/2005

授課老師: 郭鴻基

請務必依序作答, 不要讓老師在整張考卷找尋失序的答案; 題目很多, 但是其中有許多是平常作業題目。

Close Book Exam., 共計140分, 最後標準化成100分

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1 (20分)

(a) (5分) Write down the Taylor expansions of e^x and e^{-x}. Draw the schematic diagrams of e^x and e^{-x}.

(b) (5分) What are the linear and nonlinear regimes of the e^x and e^{-x}. Why is e^{-1} scale used in the science? Why is not e^{-2} or e^{-3} used? What is the e^{-1} scale for the functions of e^{-kt} and $e^{-x^2/n}$?

(c) (5分)Use the Taylor expansions and the definition of differentiation to prove that $de^x/dx = e^x$.

(d) (5分) Use the Taylor expansions and the definition of differentiation to prove that $d\ln x/dx = 1/x$.

2 (15分)

(a) (5分) Find the eigenvalues and eigenvectors of the following matrix A.

$$A = \begin{pmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{pmatrix}$$

(b) (5分) In a evolution process, let N mean "No mutation" and Y "mutation". Let the evolution transition probability from one generation to the next be 0.4 for N to N (thus, 0.6 for N to Y), 0.2 for Y to N (0.8 for Y to Y). If there is no mutation in the first generation, what are the percentages of mutation and no mutation 2 generations after?

(c) (5分)What are the most likely percentages of mutation and no mutation many generations after?

3 (20分)

(a) (5分) If $\mathbf{A}$ is a skew-symmetric matrix, and the governing equations are

$$\frac{d\mathbf{u}}{dt} = \mathbf{A}\mathbf{u}.$$

Write symbolically and transform the above equation into a decoupled system with the eigenvalue diagonalized matrix $\mathbf{D}$(以矩陣符號, 利用特徵向量矩陣將上面矩陣方程轉換成含$\mathbf{D}$對角線矩陣的不偶合矩陣方程式。)

(b) (5分) Write the nondimensionalized spring equation

$$\frac{d^2x}{dt^2} = -x,$$

in the matrix form

$$\frac{d\mathbf{u}}{dt} = \mathbf{A}\mathbf{u}.$$

(c) (5分) Find the eigenvalues and eigenvectors of matrix A. Discuss the phase and periodicity of the eigenvectors.

(d) (5分) Transform the nondimensionalized spring matrix equation into the eigenvector and find the decoupled system.

4 (10分) x_1是血液以及骨頭內含鉛濃度,x_2是組織部分含鉛濃度。考慮下列控制方程式

$$\frac{dx_1}{dt} = I - \alpha x_1 - k_{12}x_1 + k_{21}x_2,$$

$$\frac{dx_2}{dt} = -\beta x_2 + k_{12}x_1 - k_{21}x_2,$$

其中 I是鉛進入人體速率,$-\alpha x_1$是腎臟排除鉛速率,$-\beta x_2$是頭髮、指甲、流汗等排鉛的速率, $k_{21}x_2$以及$k_{12}x_1$爲x_1與x_2交換速率。請以方塊以及及箭頭畫圖討論表達這二方程式意義。令x_3爲血液、骨頭以及組織部分總含鉛量, 寫出x_3之方程式 (不用解方程式)。

5 (40分) Formulate or discuss the differential equations in the following occasions. You do not need to solve the equation(s).

(a) (5分) Malthusian model.

(b) (5分) Logistic Models.

$$\frac{dP}{dt} = kP(1 - \frac{P}{M}).$$

解釋爲何 M是環境承載量 (**carrying capacity** of the environment).

(c) (5分) Percentage of the task learned at any time if the rate of learning is the percentage of a task **not** learned.

(d) (5分) Let X and Y be the population of the predator (掠食者) and the prey (被掠食者) respectively. Write a nonlinear system of ODEs for dX/dt and dY/dt.

(e) (5分) 解釋下列方程式之意義,

$$\frac{dx}{dt} = x(1 - \frac{1}{2}x - y),$$

$$\frac{dy}{dt} = y(1 - y - \frac{x}{1+x}),$$

x族群的環境承載爲何? 那個族群在其族群數目增加後, 其競爭力沒有相對增加。如何由數學式子看出來?

(f) (5分) 解釋下列方程式之意義,

$$\frac{dx}{dt} = x - 2x^2 - \sqrt{x}y,$$

$$\frac{dy}{dt} = y(1 - y - x^2),$$

那個族群的環境承載較小? 那個族群較有以寡擊衆, 以一當十的競爭力? 如何由數學式子看出來?

(g) (5分) 寫出兩個族群「合作則雙贏, 分開則雙輸, 族群滅絕」的共生方程式。如何更動兩個共生族群方程式, 讓其中一個族群比較「自私」。

(h) (5分)寫出兩個族群各自滿足 Logistic equation, 彼此不直接交互作用, 但是每個族群擁有的環境承載量是取決於對方的族群數目。

6 (25分, 每題5分) Solve the following first order ODEs with y_0 as the initial condition.

(a) $dy/dt = -\alpha y$, (b) $dy/dt = 1 - y$

(c) $dy/dt = y - y^2$, (d) $dy/dt + \lambda y = f_0 e^{i\Omega t}$.

(e) Explain why the solution of the equation

$$\frac{dy}{dt} = 1 - y$$

is bounded by 1 (i.e. $y \to 1$, as $t \to \infty$). Also explain why y gets slower and slower approaching to 1.

7 (10分) Formulate the differential equations in the following occasions and to briefly explain the equation you write. [No need to solve the equation.] Suppose the rate of increase of the 矮人 population of the 鐵爐堡 proportional to the current population, with some fixed proportionality constant a (in unit of year^{-1}). At certain point, say at year t_0, the 鐵爐堡 decides to foster 獵人、戰士 immigration, which thereforth occurs at the constant level of b individuals per year (continuously, of course). It is also assumed that 地精 with c individuals came to 鐵爐堡 at year t_0 only. It is assumed that 獵人、戰士、地精 immigrants assimilate (merged) immediately and do not affect the rate a.