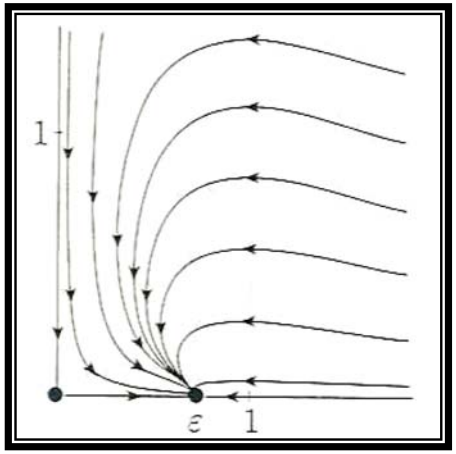
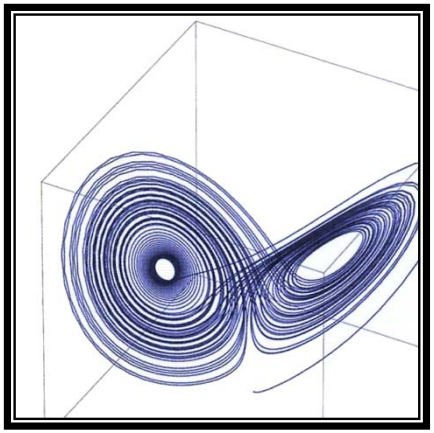
數學建模與因次分析

主講人：郭鴻基 教授



（掠食者的滅絕）

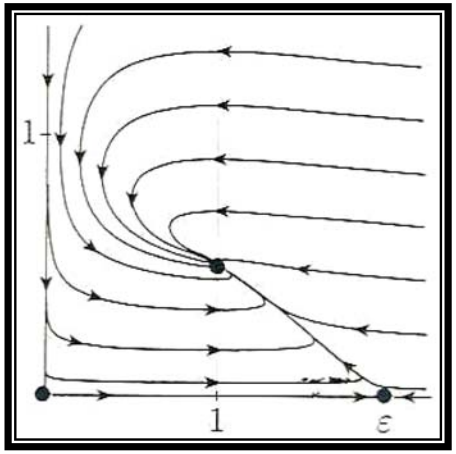


（Lorenz 吸子）

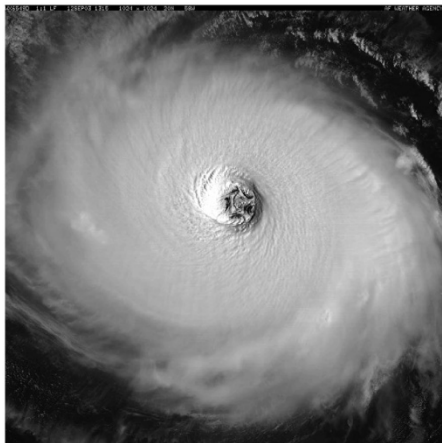
內容：

科學研究是探索未知知識的建構過程，而數學是科學的語言。隨著電腦的進步，資料大量的數位化，科學計算成為非線性科學研究的敲門磚。數學建模、科學計算、分析詮釋與驗證等過程，更是現今數學科學的典範。數學函數本身是無因次，但是科學量化卻有單位，演講將簡介因次分析，並舉例說明數學建模、科學計算、因次分析詮釋等過程。

2008/7/24 國家災害防救科技中心



（掠食者與被掠食者共存）



To derive the equations ..... or not to derive the equations that is a question !!

讀 算 寫

幾何
代數
微積分
電腦計算繪圖

+ -

加、減

線性

大題大作

x /

乘、除

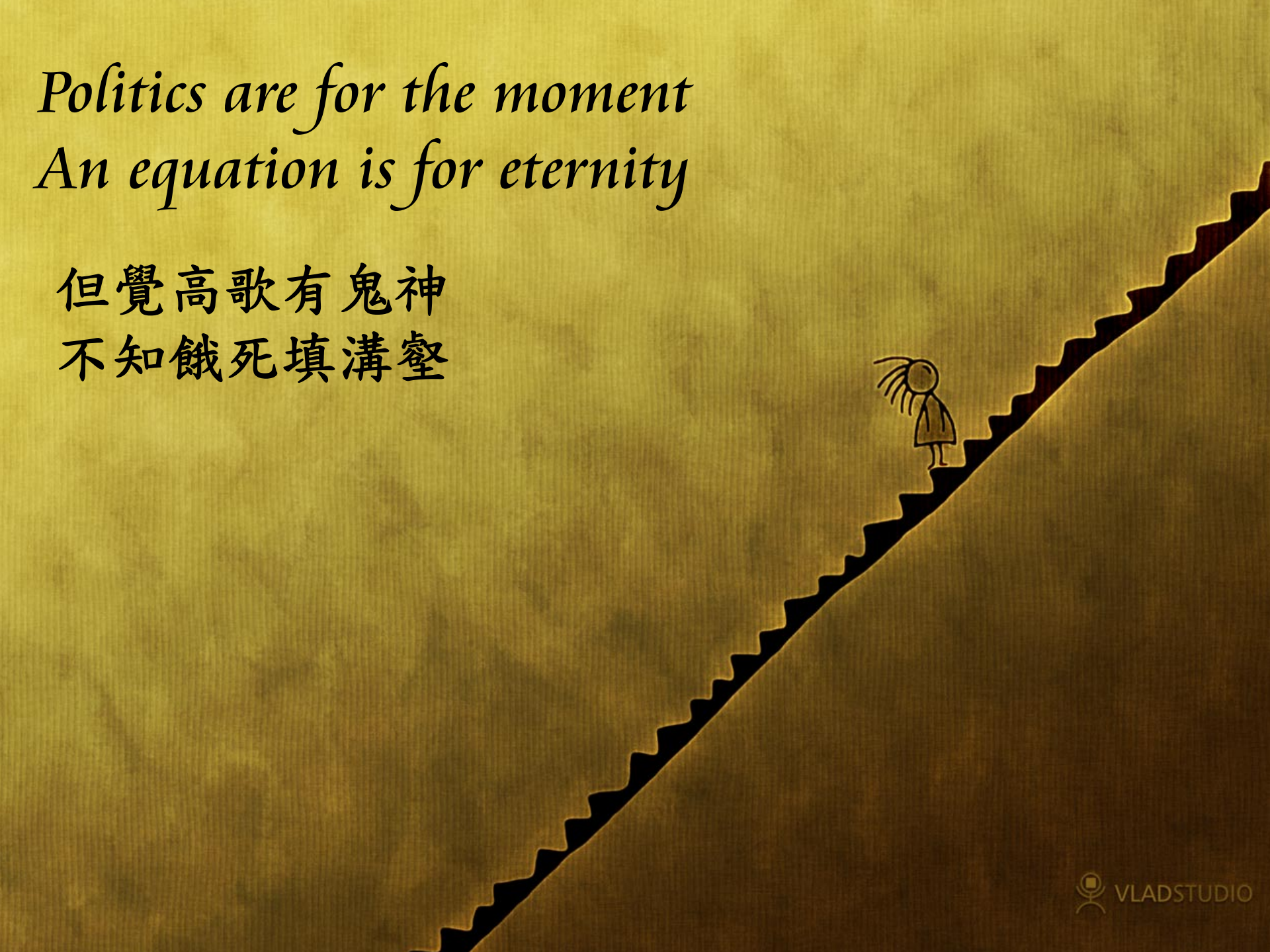
非線性

小題大作

數量化、數位化
數學化--模式--動力系統

Politics are for the moment
An equation is for eternity

但覺高歌有鬼神
不知餓死填溝壑



數學科學

Formulation　　微分、差分方程式

Solution / Analysis　分析、解

Interpretation　科學詮釋

中階課程：微分方程(ODE,PDE)
統計、線性代數
程式、計算與繪圖

數量化、數位化

數學化--模式--動力系統

“Six monkeys, set to strum unintelligently on typewriters for millions of years, would be bound in time to write all the books in the British Museum.” Huxley

君子致用在乎經邦，經邦在乎立事，立事在乎師古，師古在乎隨時。必參古今之宜，
窮終始之要，始可以度其古，中可以行於今。通典

共**49**個字，假設中文常用字爲**1000**字，共有**10^147**個選擇

地球歷史 **10^18** sec
10^10 一百億隻猴子在打字，假設每秒鐘打一萬字 **10^4**，

10^10*10^18*10^4 = 10^32

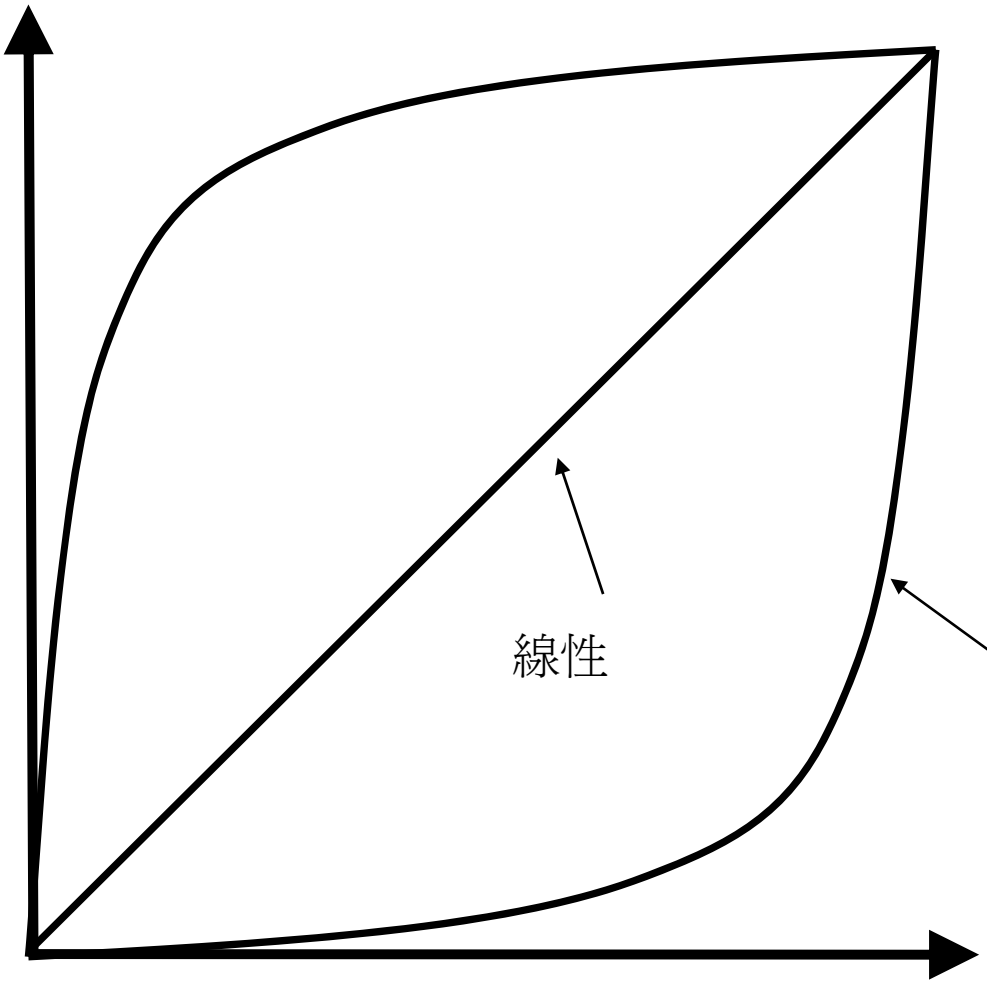
10^32/10^147=10^(-115) ~ 0 機率爲零，不可能的巧合！

研究學問是苦心孤詣的事業！ 不要人云亦云！

一分耕耘 ， 一分收穫？？

函數 **Function**

(因果）



y：成就

$$\frac{dy}{dx} > 0 \Rightarrow \frac{d^2y}{dx^2} = ?$$

x：努力

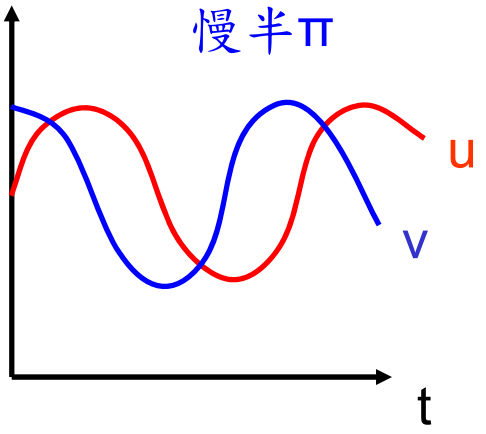
動力系統

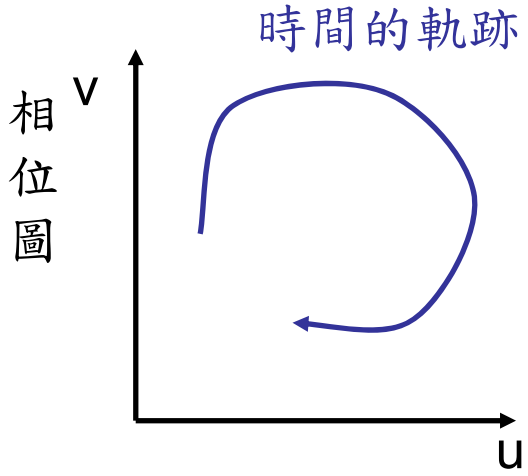
$$\frac{d\mathbf{u}}{dt} = f(\mathbf{u}, \gamma_i)$$

時間變化謂之動力　　變數　　許多外在及內在控制參數

$$\int_0^{2\pi} \cos t \sin t dt = 0$$
$$u \quad v$$

$$\overline{uv} = 0$$





Cos 和 **Sin** 零相關、不來電！

推背圖：前知三百年，後知三百年

可以解釋〝已知〞，可以預測〝未來〞

相位圖

微積分數學

$$u = u(x, y)$$

Chain Rule(連鎖律)

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

偏微分

例如: $\frac{\partial x^2 y}{\partial y} = x^2$

只對y變數微分，不改變x變數

你快樂嗎？一個簡單的生涯規劃動力系統

u：快樂指數

x:考試作業量

y:玩魔獸的時間

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

天縱英明的資優生　<0　>0　<0　<0　>0

考試越少越不快樂，

玩魔獸的時間越多越不快樂

$\frac{\partial u}{\partial x}$ >0 考試越多越快樂

$\frac{\partial u}{\partial y}$ <0 電動越玩越不快樂

→ 人的個性

$\frac{dx}{dt}$

$\frac{dy}{dt}$

→ 人的境遇

個性+境遇=人生

相形不如論心

論心不如則術

形不勝心

心不勝術　荀子非相

HIV Modeling

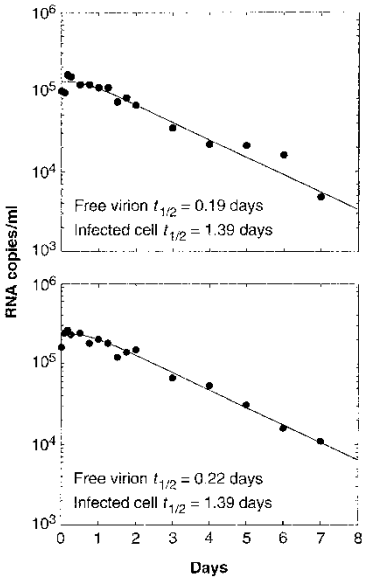
Perelson and Nelson (1999)

$$\frac{dV}{dt} = P - cV,$$

藥物治療

$$\frac{dT}{dt} = kT_0V - \alpha T,$$

$$P = N\alpha T.$$



觀察病人服藥後反應決定C

Figure 4.5. Log of plasma concentrations (copies per mL) of HIV-1 RNA (circles) for two representative patients (upper panel, patient 104; lower panel, patient 107) after *ritonavir* treatment was begun on day 0. The solid line is a nonlinear least square fit to the data. HIV-1 RNA level is an easier measure of HIV virions since each HIV virion contains two RNA molecules. (See exercise 5 for more details.) (From Perelson et al. [1996], used by permission of Alan S. Perelson.)

$$P(t_0) \cong cV(t_0) \sim 2 \times 3 \times 10^5 \quad (1/(day \cdot ml))$$

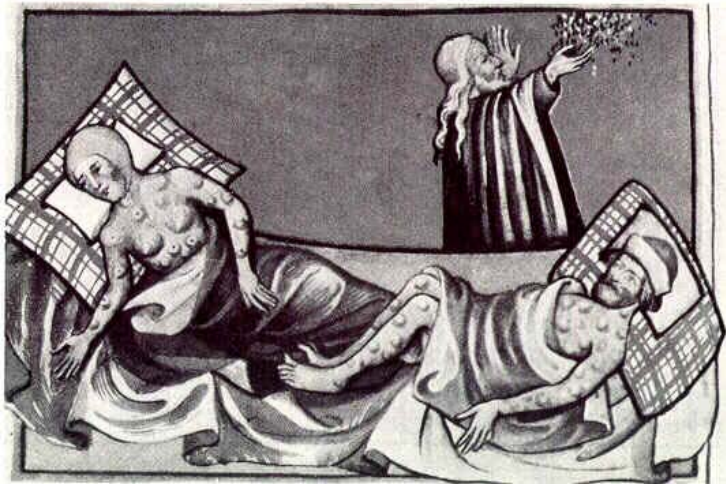
V: number of virions
p: rate of production of new HIV virions
c: clearance rate for the virions in the plasma
T: infected target cells in unit volumn
: non-infected cells in unit volumn
N: 被感染細胞在其生命期內產生的病毒數目
k: 正常細胞被病毒感染率

Early and aggressive therapeutic intervention is necessary if a marked clinical impact is to be achieved.

何大一雞尾酒療法

Epidemics

- **Epidemics: *epi* “upon” and *demos* “the people” , i.e., “upon the people”**
- **An epidemic is the occurrence in a community or region of cases of an illness, specified health behavior, or other health-related events clearly in excess of normal expectancy; the community or region, and the time period in which cases occur, are specified precisely** (Last JM, ed. A Dictionary of Epidemiology. New York: Oxford University Press, 1995)



The “***Black Death***" of 1347–51

SIR Model

$$\frac{dS}{dt} = -\beta SI$$

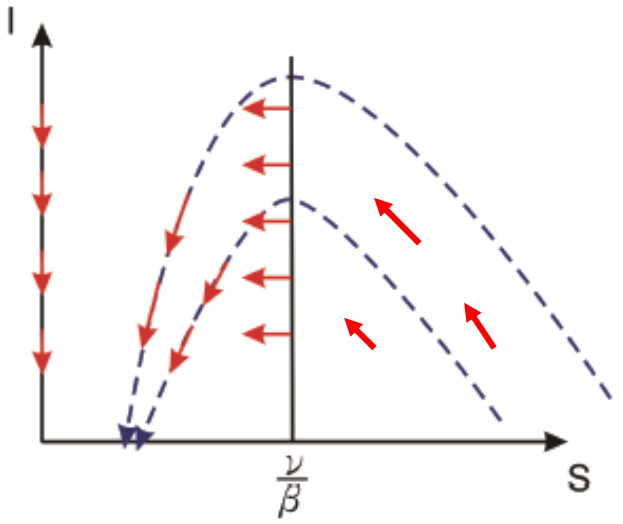
$$\frac{dI}{dt} = +\beta SI - \nu I = (\beta S - \nu)I$$

$$\frac{dR}{dt} = +\nu I$$

ν Recovery Rate

β Infection Rate

No Death in the model



$S = \frac{\nu}{\beta}$有 null cline

Forecast and control of epidemics in a globalized world

PNAS vol.101 no.42

Hufnagel†, Brockmann, and Geisel

演講者: 陳怡文　日期: 2007/12/18 指導教授: 郭鴻基 老師

Use the SIR model with the stochastic forcing from international aviation network to simulate the spread of the SARS, and to explore the strategy for the disease control.

WHY 無因次化？？

數學是無因次　上帝創造人類 人類創造單位

20分 與 100分 ？ 快慢？ 大小？

標準化　Top 1%

手術死亡3人　與手術死亡100人 何者為佳？

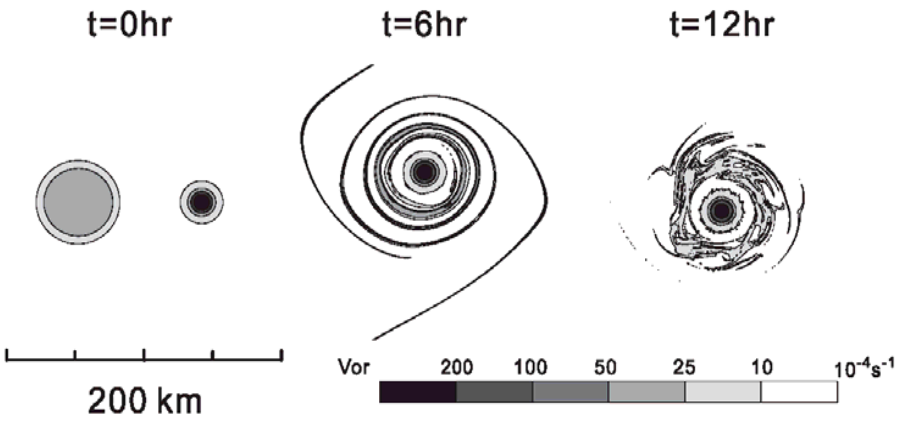
線性與非線性

Eddy turn over time

Mathematical elegance and analysis

Important parameter(s)

因次與無因次相輔相成



颱風雙眼牆動力

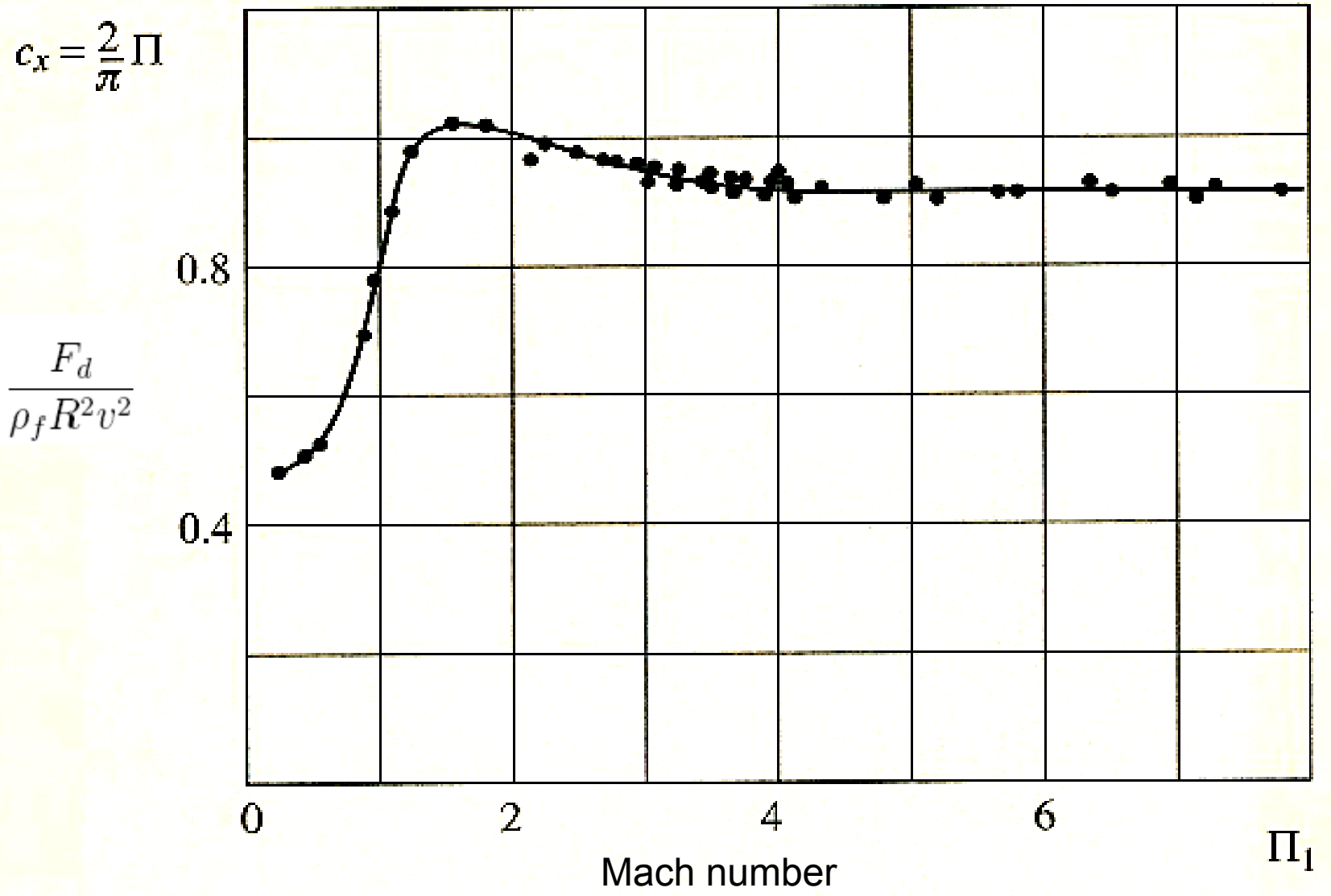
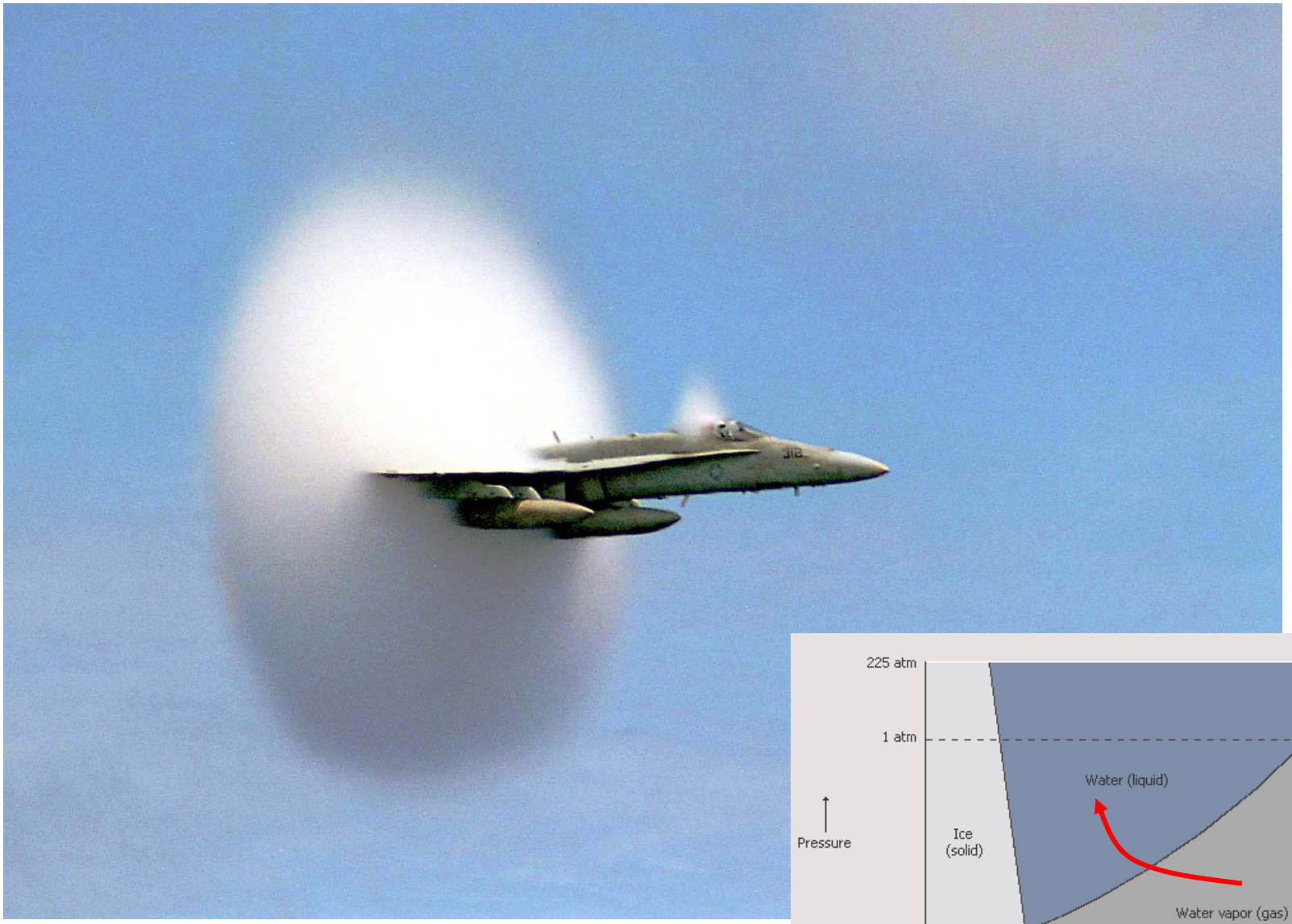
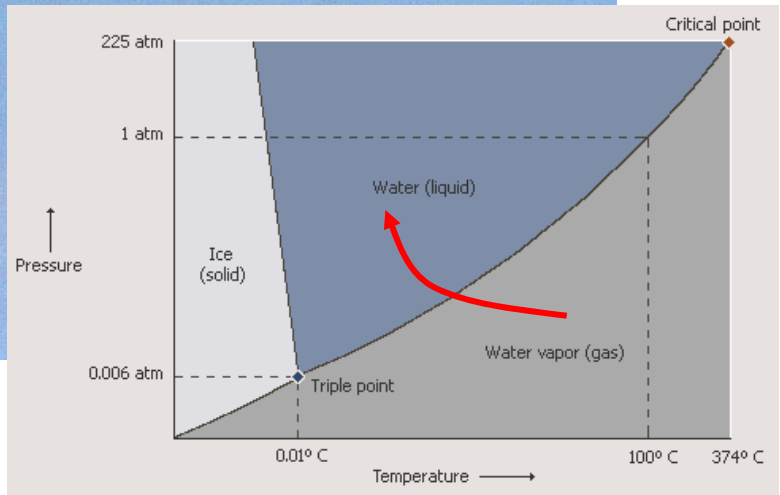


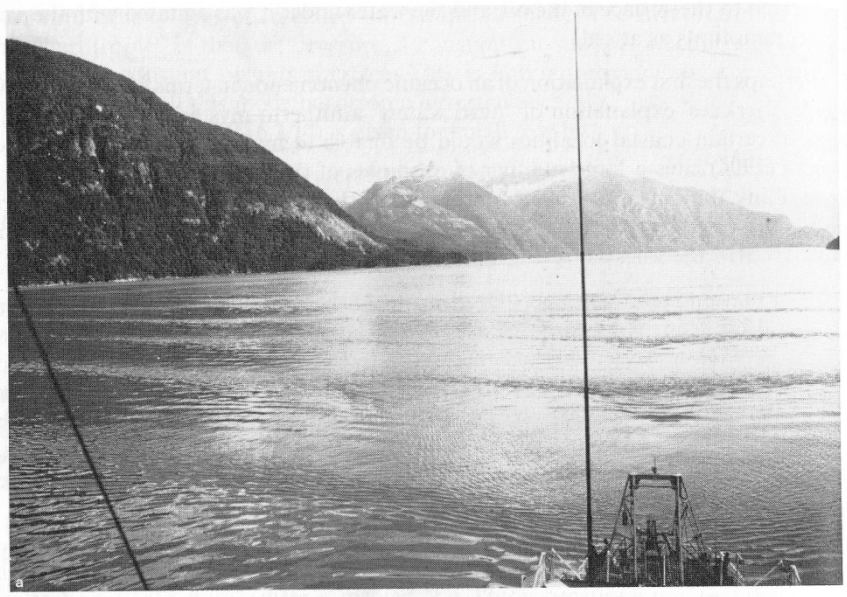
Figure 1.11. The dimensionless drag on a sphere, Π (times $2/\pi$), as a function of the dimensionless governing parameter $\Pi_1 = U/c$, the Mach number (Chernyi 1961). The quantity Π approaches a constant for large values of Π_1.

Sonic Boom





Adiabatic Sound



Dead-water near the coast

Wave drag problem

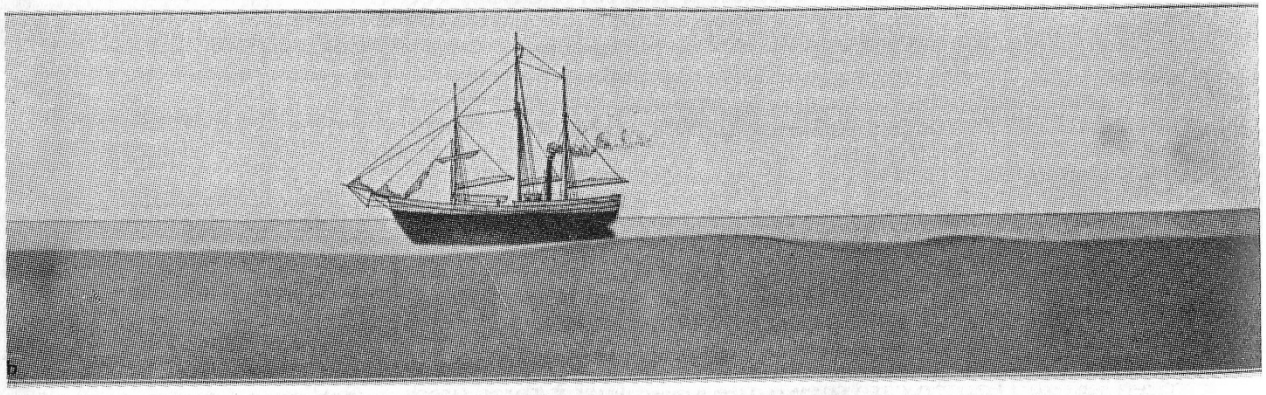
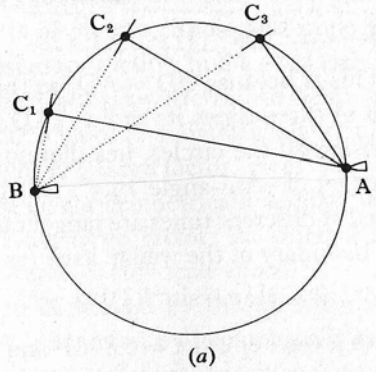


Fig. 6.2. (a) Surface "slicks" showing the presence of internal waves in the wake of a ship in Bute Inlet, British Columbia. The vessel was traveling at 0.5 m s^{-1} in a surface layer of almost fresh water only slightly deeper than its 3.4 m draft. The internal waves caused horizontal motion at the surface that affects the ripple pattern and so renders the internal wave pattern visible at the surface during calm conditions. [Photo courtesy of Defence Research Establishment Pacific, Victoria, British Columbia.] (b) A laboratory experiment [from Ekman (1904)], showing internal waves being generated by a model ship. The tank is filled with two fluids of different density, the heavier one being dyed to make the interface clearly visible. The model ship (the superstructure of the "Fram" has been drawn in subsequently) is towed from right to left, causing a wake of waves on the interface.



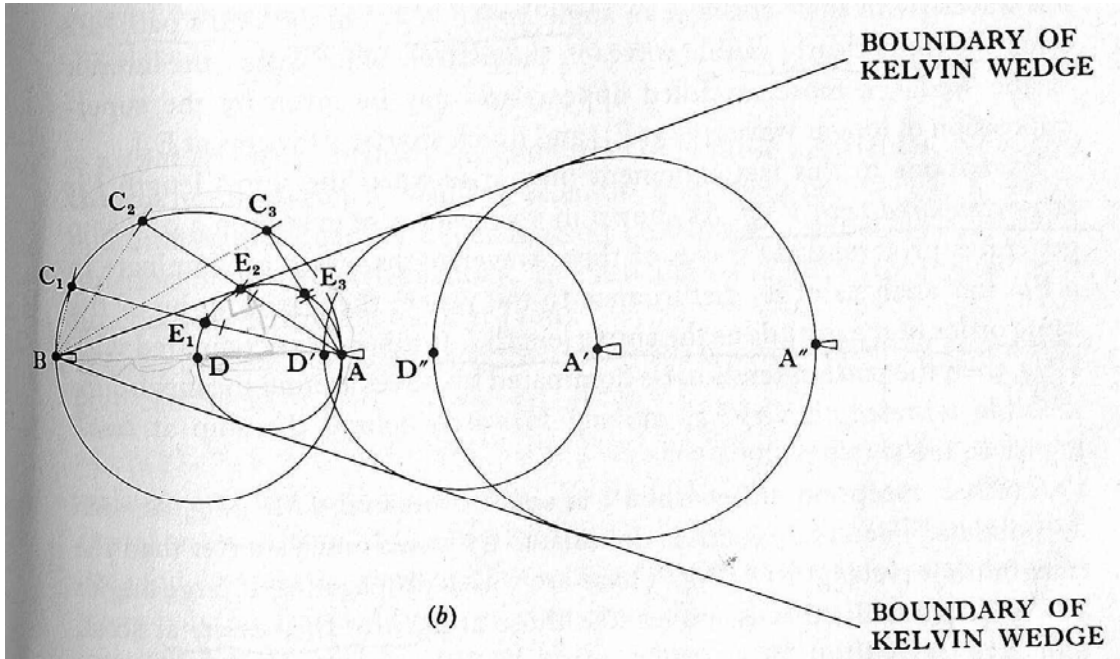


Figure 68. Waves generated in deep water by a ship B. Case (*a*): positions C_1, C_2, C_3 of any waves generated t_g seconds ago (when the ship was at A) if their energy had travelled a distance ct_g. Case (*b*): the real positions E_1, E_2, E_3 of the same waves, taking into account that their energy has only travelled a distance $\frac{1}{2}ct_g$. At each, the dependence of wavelength on direction of emission, as inferred from equation (183), is shown. The circle with diameter AD is the locus of all such waves. Other such circles, with diameters A′D′ and A″D″, are where waves generated when the ship was at A′ and A″ are now to be found. All such circles lie within the Kelvin wedge of semi-angle (184).

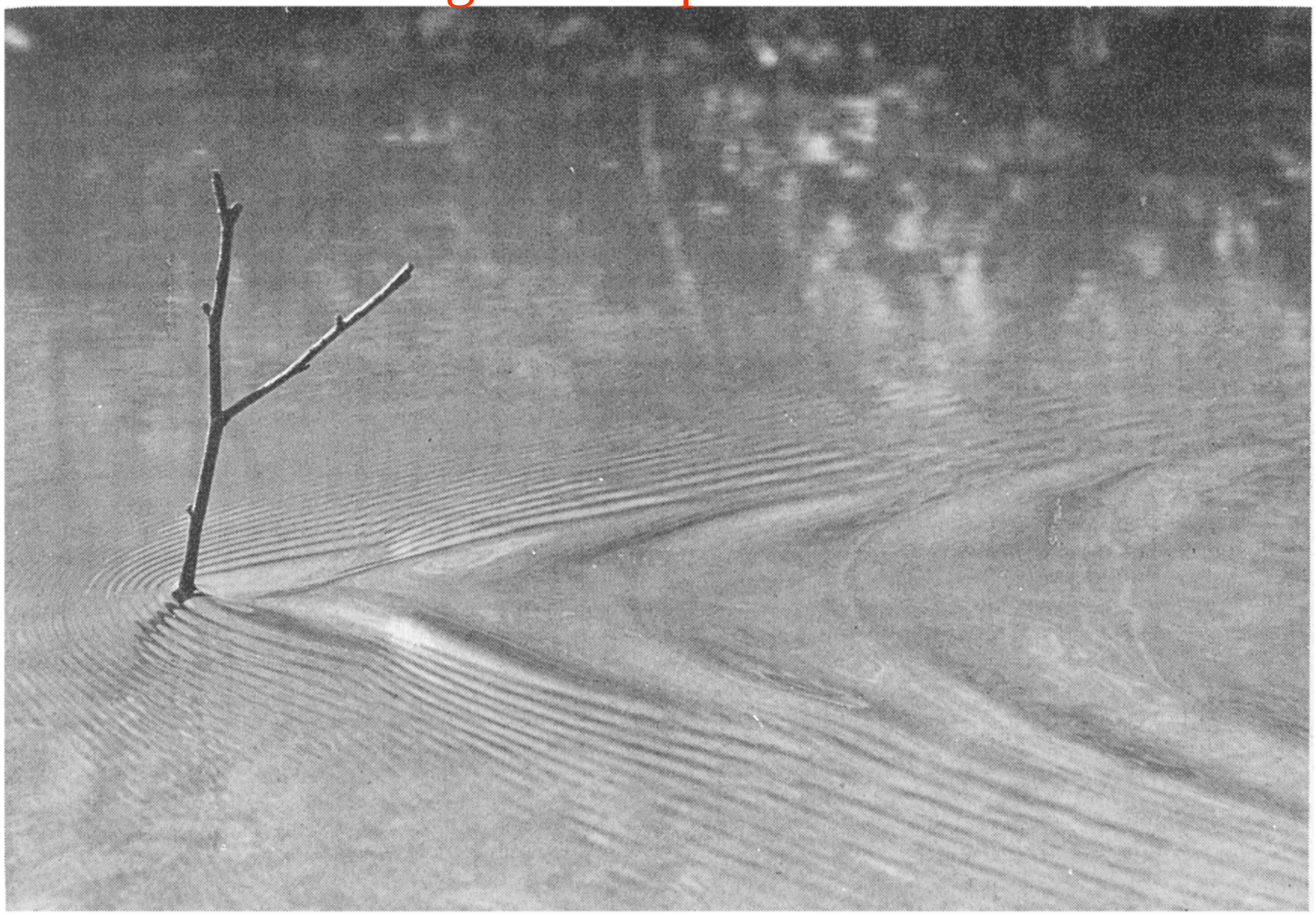
Kelvin edge deep water waves

< Ship-wave pattern >

$$2 * \sin^{-1}\left(\frac{1}{3}\right) = 2 * 19.5^{o}$$

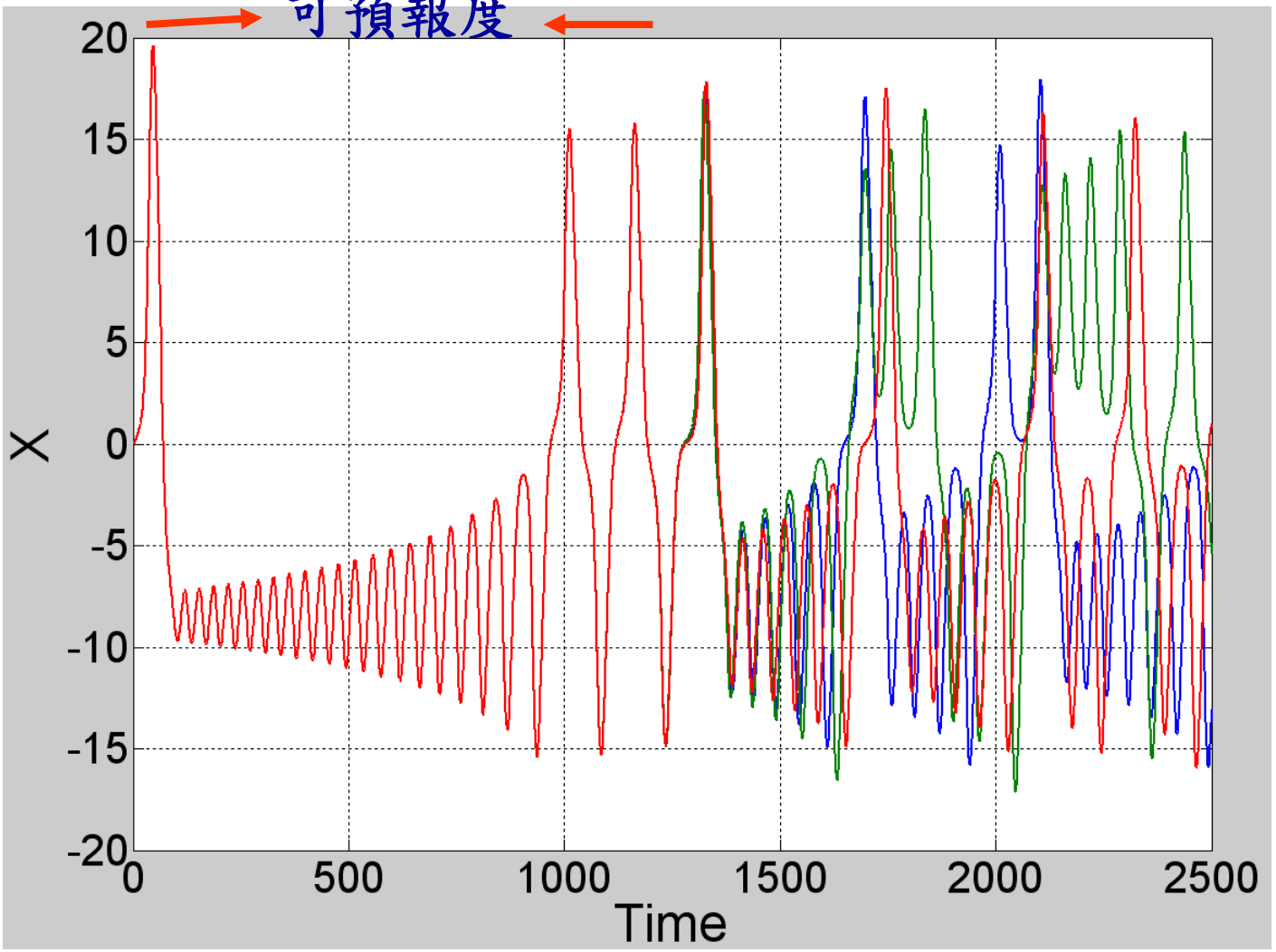
[*Courtesy of Aerofilms Ltd.*]

Kelvin wedge deep water waves



[*From V. A. Tucker, "Waves and Water Beetles,"*
Physics Teacher **9**, 10-14 19 (1971), Fig.3
(Copyright 1971 by American Assoc. Phys. Teachers)]

Time Series – Initial Condition Slightly Difference



(x,y,z)=(0,1,1)

(x,y,z)=(0,1.001,1)

(x,y,z)=(0,0.999,1)

蝴蝶效應 Butterfly Effect Chaos 混沌

混沌　　非線性　　$y = x^{30}$　　精確度有限 非線性

$$0.02 \begin{cases} x = 0.99 \\ x = 1.01 \end{cases} \qquad 0.61 \begin{cases} y \approx 0.74 \\ y \approx 1.35 \end{cases}$$

預報能力的喪失！！

“Sensitivity dependence on initial condition.”

H Poincare

$$\frac{d\,x}{d\,t} = x(1-x) - h$$

h: constant rate of population harvested.

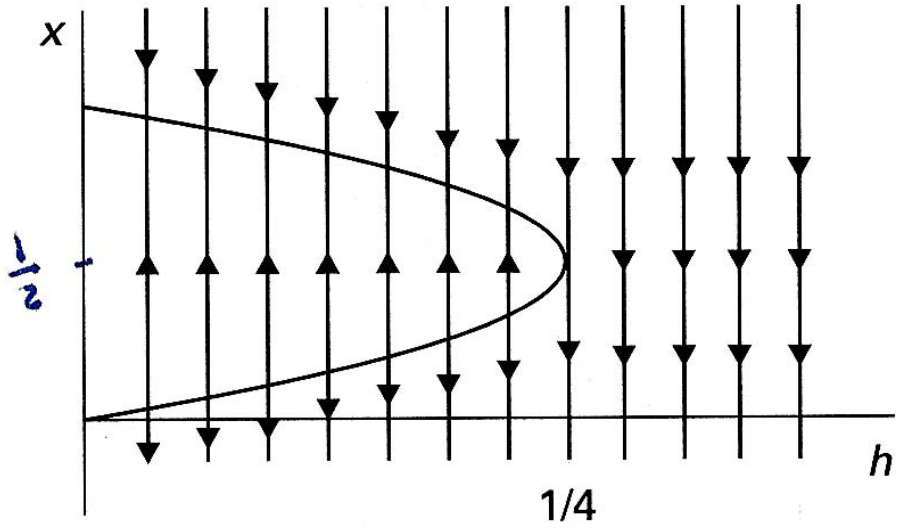


Figure 1.7 The bifurcation diagram for $f_h(x) = x(1 - x) - h$.

Small changes in harvesting rate can lead to disastrous changes in population has been observed many times in real situations on earth.

$$\frac{dP}{dt} = \gamma P\left(1 - \frac{P}{N}\right) - h$$

N 環境承載

γ 成長率

$$P^* = \frac{P}{N} \quad t^* = \gamma t \quad h^* = \frac{h}{N\gamma}$$

h 捕獲率

$$\frac{dP^*}{dt^*} = P^*\left(1 - P^*\right) - h^*$$

$$h < \frac{1/4\ N}{1/\gamma}$$

補獲率在一個生長期內不可以超過環境承載1/4

Buckingham's Pi Theorem

n variables can always be combined to form
Exactly (n- r) independent nondimensional variables, where
R is the rank of the dimensional matrix.

Eg. T, M, L, g are the 4 variable in the simple pendulum problem

kg, m, s are the rank 3 dimensional unit

4-3 =1 nondimensional variable $T \sim \sqrt{\frac{l}{g}}$

Similar result can be derived from Newton's mechanics or
from the fact that the difference of potential energy and kinetic energy over a period will be minimized; the Hamilton principle.

$$\Pi = \frac{r_f}{E^{1/5} t^{2/5} \rho_0^{-1/5}}$$

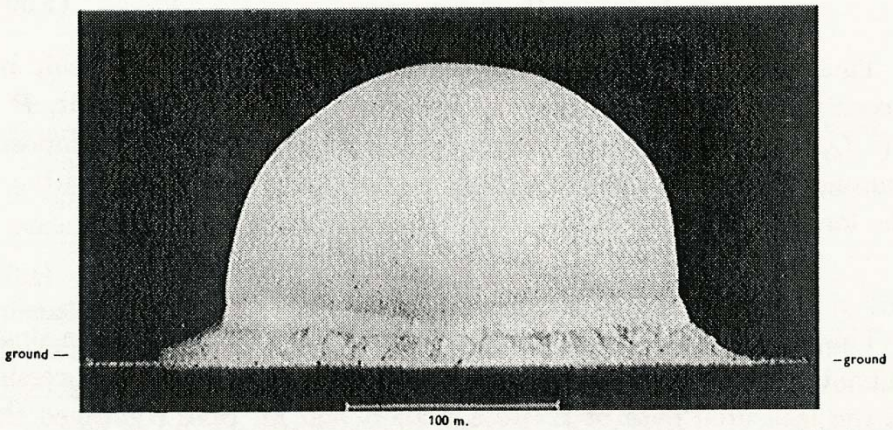


Figure 1.5. A photograph of a fireball 15 ms after an atomic explosion on the ground illustrates the spherical symmetry of the phenomenon and the sharp boundary of the perturbed region (Taylor, 1950a, b, 1963).

G.I. Taylor 1950

Symbol	*Definition*	*Representative value or first guess*
R	radius of wavefront	10^2 m
t	time	10^{-2} s
p_0	ambient pressure	10^5 Pa
ρ_0	ambient density	1 kg m^{-3}
E	energy released	10^{14} J

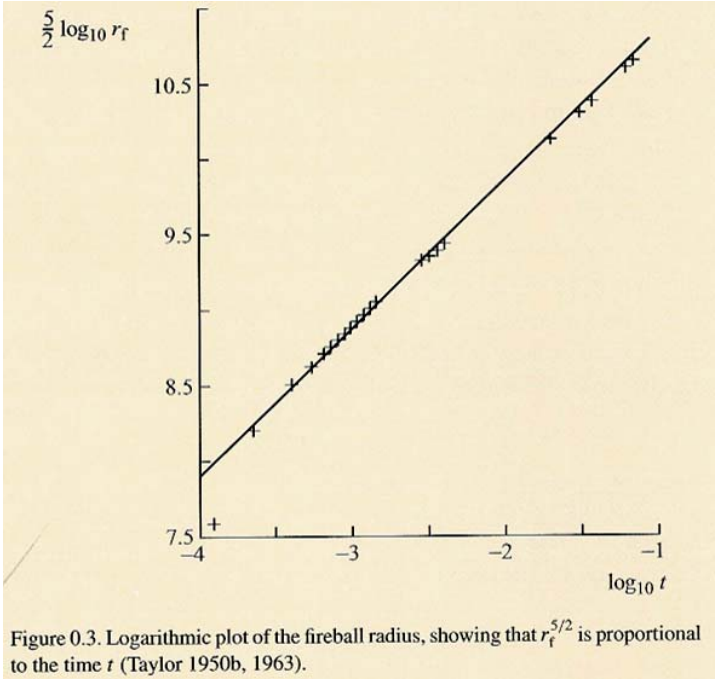
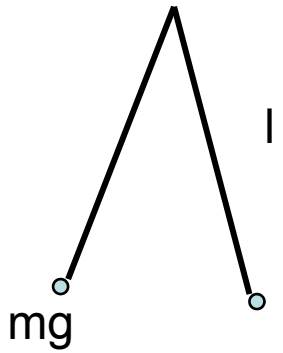


Figure 0.3. Logarithmic plot of the fireball radius, showing that $r_f^{5/2}$ is proportional to the time t (Taylor 1950b, 1963).

原子彈能量～ 10^{14} J



$$T \sim \sqrt{\frac{l}{g}} \qquad V \sim \frac{l}{T} \sim \sqrt{l}$$

Walking speed

$$T \sim \sqrt{\frac{I}{mgr}} \sim \sqrt{\frac{ml^2}{mgl}} \sim \sqrt{l}$$

$$T \sim \sqrt{\frac{I}{L_{max}}} \qquad T \sim \sqrt{\frac{l^3 l^2}{l^2 l}} \sim l$$

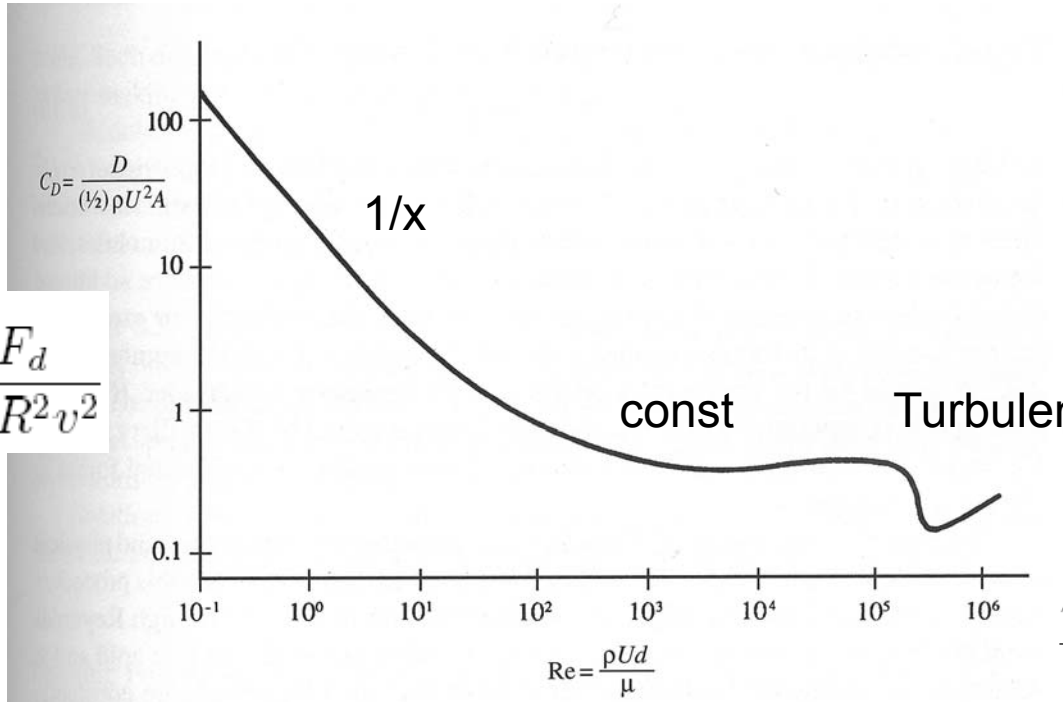
Runner speed is independent of leg length

$I = l^2 m/3$ the moment of inertia

Force is proportional to muscle area
Mass m is proportional to volume

L_{max} is the torque (force * length)

$$L_{max} \sim F_m l \sim l^2 l,$$



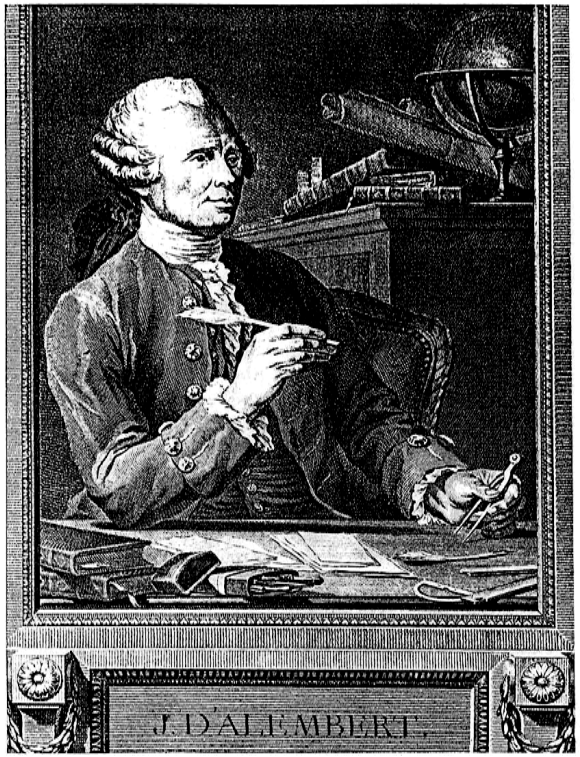
$$\frac{F_d}{\rho_f R^2 v^2}$$

$$\frac{vR}{\nu} = \mathbf{Re}$$

Figure 8.2 Drag coefficient for a sphere. The characteristic area is taken as $A = \pi d^2/4$. The reason for the sudden drop of C_D at Re $\sim 5 \times 10^5$ is the transition of the laminar boundary layer to a turbulent one, as explained in Chapter 10.

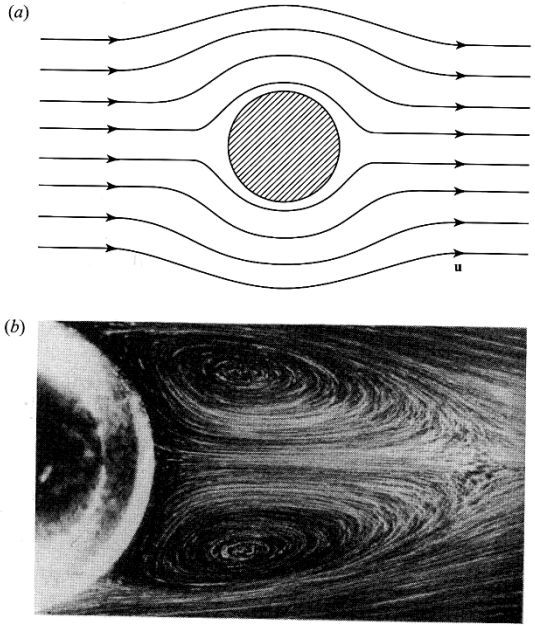
F_d	$[M][L][T]^{-2}$	drag force;
ρ_f	$[M][L]^{-3}$	density of fluid
R	$[L]$	radius of sphere
v	$[L][T]^{-1}$	speed
ν	$[L]^2[T]^{-1}$	kinematic viscosity.

$$t_v >> t_\nu, \quad \frac{R}{v} >> \frac{R^2}{\nu}, \quad or \ \frac{vR}{\nu} = \mathbf{Re} << 1.$$



1717-1783

D'Alembert Paradox



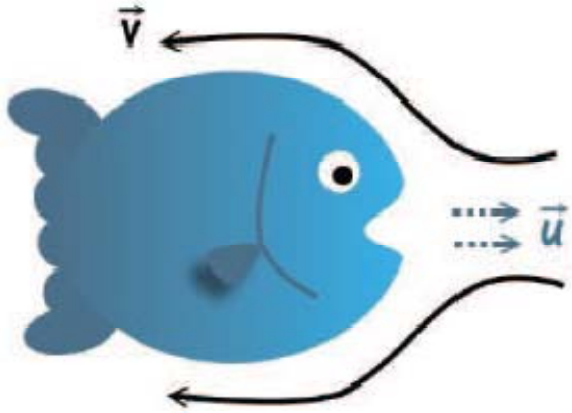
$$t_v >> t_\nu, \quad \frac{R}{v} >> \frac{R^2}{\nu}, \quad or \ \frac{vR}{\nu} = \mathbf{Re} << 1.$$

D'Alembert Solution of the Wave Equation
[f(x + ct) and f(x-ct)]

Re small viscosity important
Re large viscosity unimportant

Atmospheric Motion first expressed mathematically
(Won the 1746 Berlin Academy's Award; aqua-planet
Endorsement of Euler)

Solar and Lunar Force Cause the Atmopheric Motion



Power = Force x velocity

Force = Density x velocity^2 x Area

據觀察, 魚游向前的速度約爲背景速度的1.5倍, 爲什麼?

$$cu^3 \frac{l}{u-v} = E \text{ (power } \times \text{ time = energy)}$$

$$\frac{dE}{dt} = 0 \text{ 魚最省力}$$

$$\Rightarrow \frac{3u^2}{u-v} + \frac{u^3}{(u-v)^2} = 0 \Rightarrow u = \frac{3}{2}v$$

Fig. 8.9. Vortices trailing form the wingtips of a Boeing 727. Figure courtesy of NASA.

Wake Turbulence

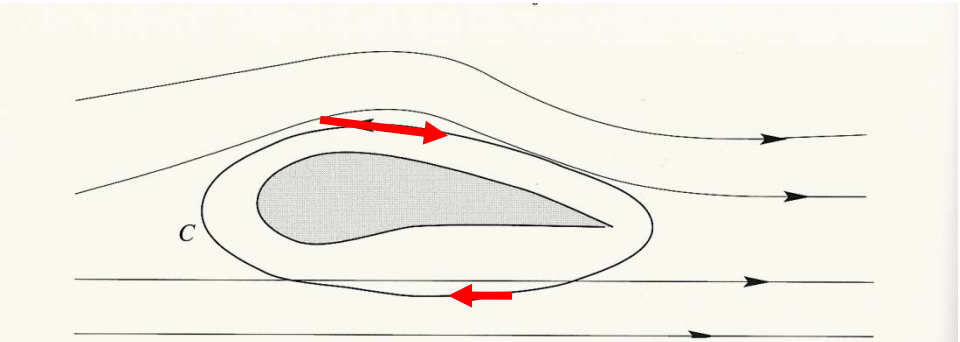


Fig. 8.10. Sketch of the flow along an airfoil. The wing is shown in grey, the contour C is shown by the thick solid line.

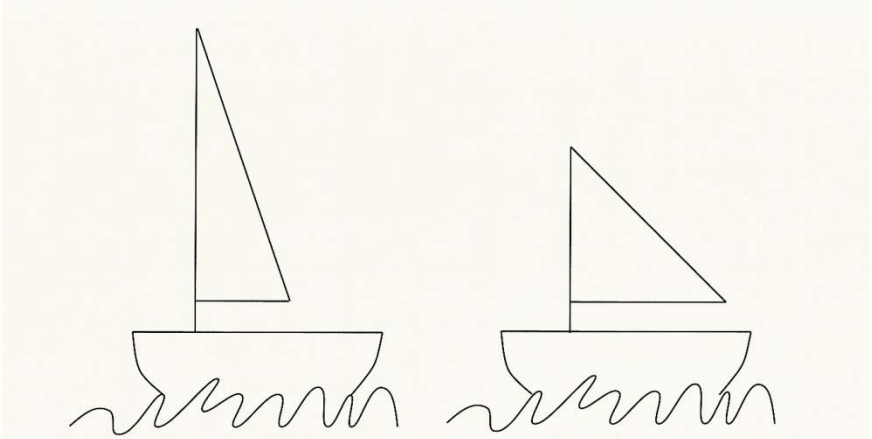
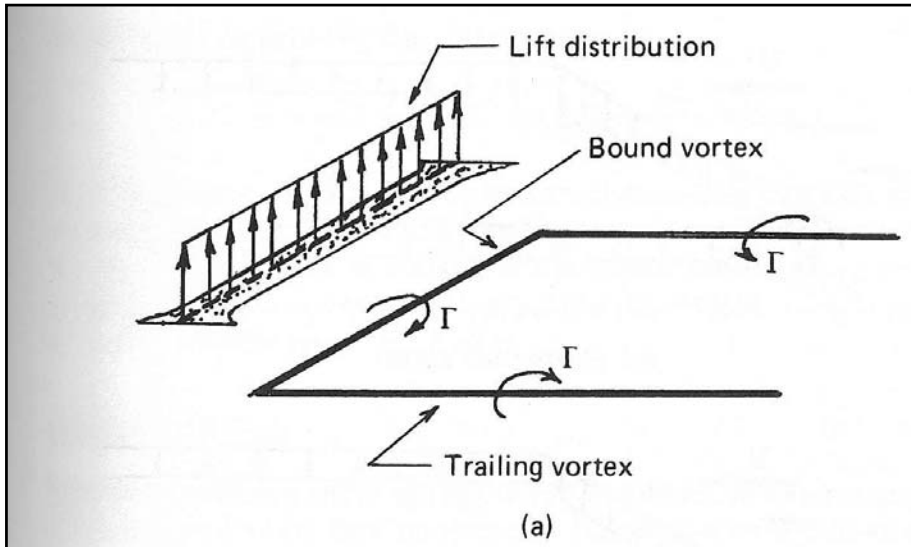


Fig. 8.12. Two boats carrying sails with very different aspect ratios.

Biomath



Lanchester

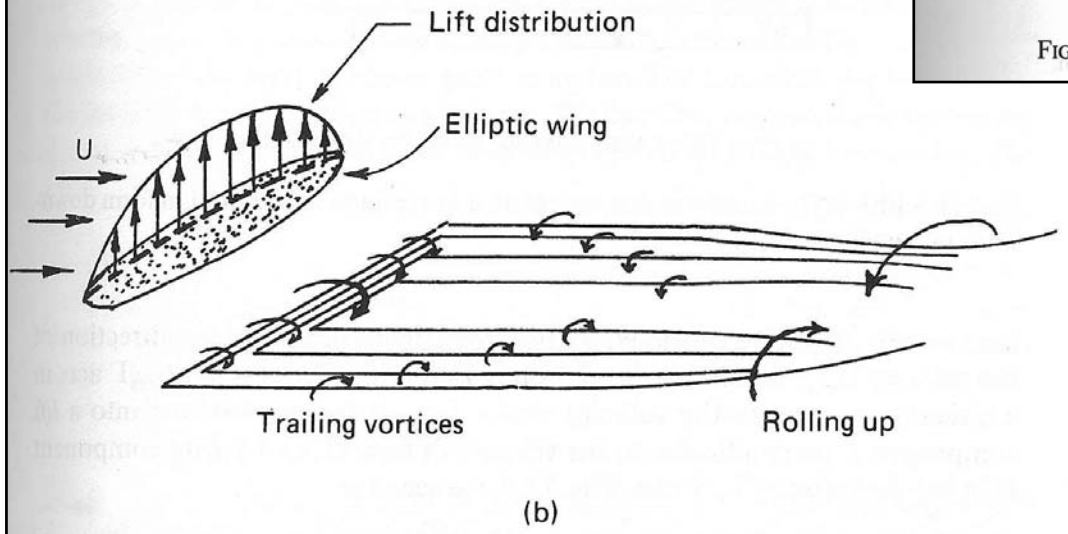


FIGURE 3.10:2 (a) A horse-shoe vortex representing a wing with a uniform lift distribution. (b) Lift distribution on an elliptic wing.

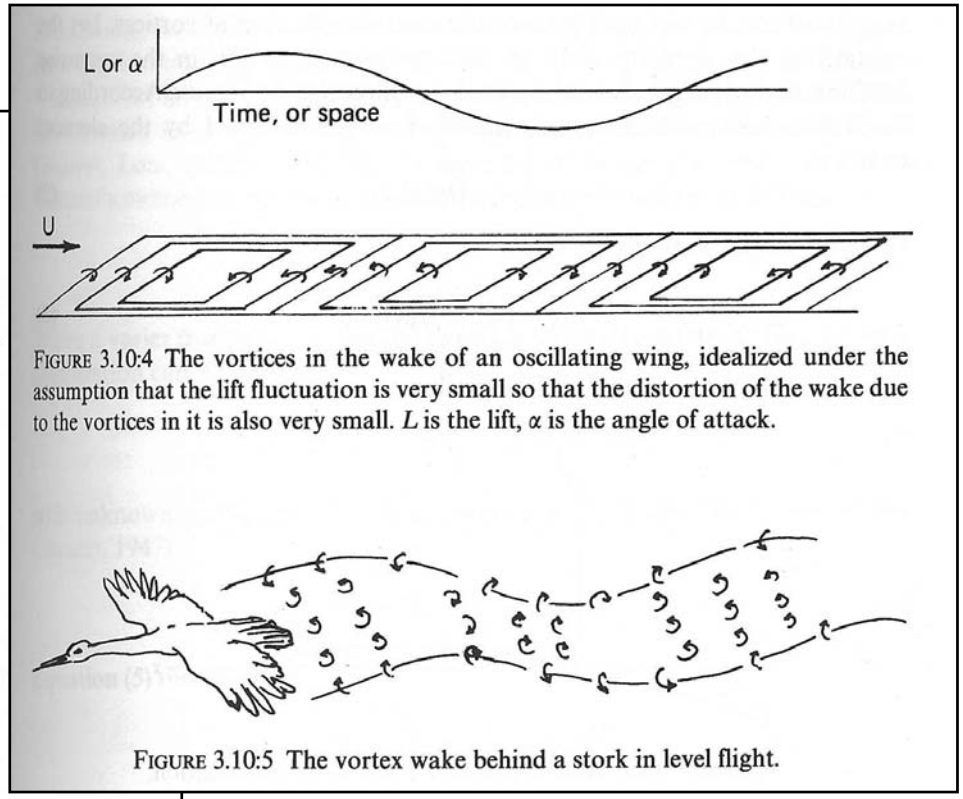


FIGURE 3.10:4 The vortices in the wake of an oscillating wing, idealized under the assumption that the lift fluctuation is very small so that the distortion of the wake due to the vortices in it is also very small. L is the lift, α is the angle of attack.

FIGURE 3.10:5 The vortex wake behind a stork in level flight.

Y. C. Fung

$$F = C_L \, \rho_{air} \, v^2 \, S$$

$$W = \rho_f \, V \, g = \rho_f \, S^{1.5} \, g$$

$$F = W$$

$$S^{0.5} = \frac{C_L \, \rho_{air} \, v^2}{\rho_f \, g}$$

$$W = \rho_f \, g \, S^{1.5} = \frac{{C_L}^3 \, \rho_{air}^3 \, v^6}{\rho_f^2 \, g^2}$$

v: speed

V: volumn

S: area

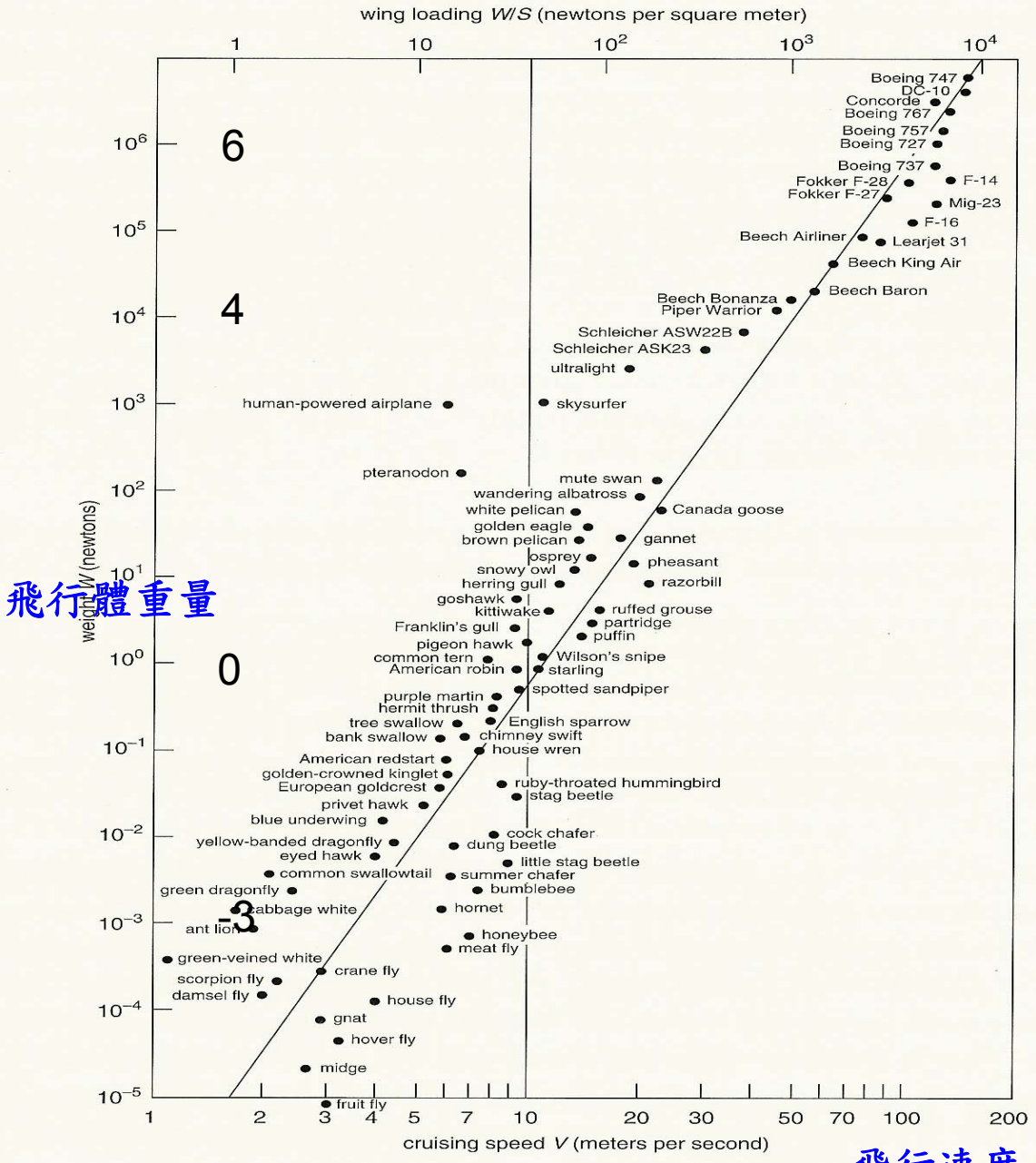


Fig. 2.2 The weight of many flying objects (vertical axis) against their cruising speed (horizontal axis) on a log–log plot. This figure is reproduced from reference [106] with permission from MIT Press.

CHINA
CHINA
CHN

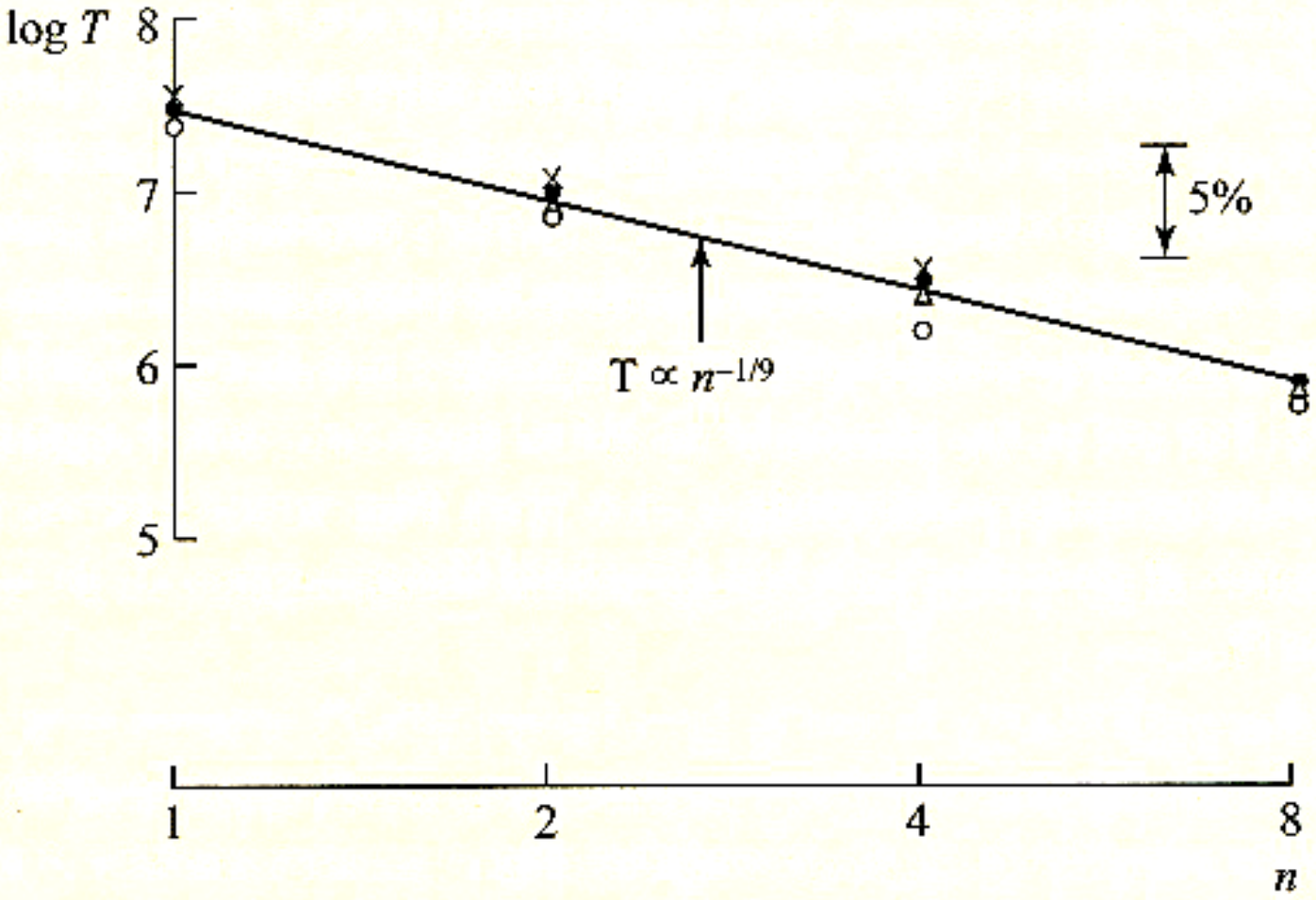


Figure 1.13. The −1/9 power-law dependence of the rowing time T on the number of oarsmen n (solid line). This may be compared with racing times over 2000 m, all at calm or near calm conditions: Δ, 1964 Olympics, Tokyo; ●, 1968 Olympics, Mexico City; ×, 1970 World Rowing Championships, Ontario; ○, 1970 Lucerne International Championships. After McMahon (1971).

$$F \sim \rho v^2 l^2$$

假設： G, A 為常數

A:strength of oarsperson

$$P \sim \rho v^3 l^2$$

$$\boxed{G \sim \frac{l^3}{n}}$$

$$l^2 \sim G^{2/3} n^{2/3}$$

$$P = nA \sim \rho v^3 l^2 \sim \rho v^3 G^{2/3} n^{2/3}$$

$$\boxed{v \sim n^{1/9}}$$

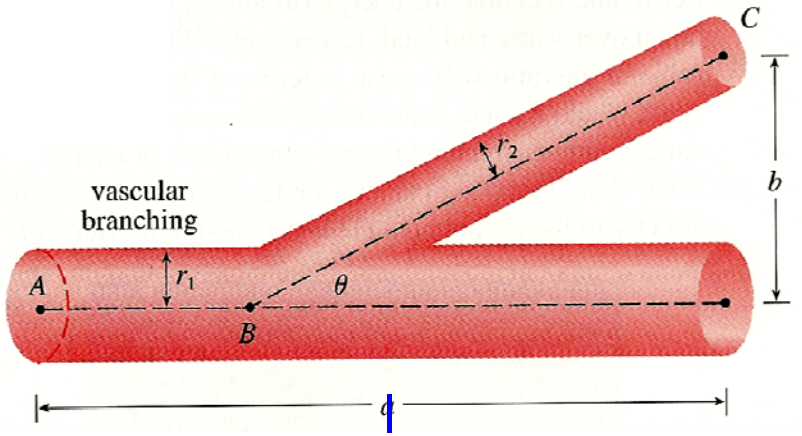
$$-2\pi r l \mu \frac{dv}{dr} = \triangle p \pi r^2$$

$$v(r) = \frac{\triangle p}{4 l \mu} \left(r_0^2 - r^2 \right)$$

$$I = \int_0^{r_0} v \, 2\pi r dr = \frac{\pi}{8} \frac{\triangle p}{\mu} \frac{r_0^4}{l}$$

$$\triangle p = I \left(\frac{8}{\pi} \frac{\mu l}{r_0^4} \right)$$

$$V = IR, \quad R = f \left(\frac{cl}{r_0^2} \right)$$

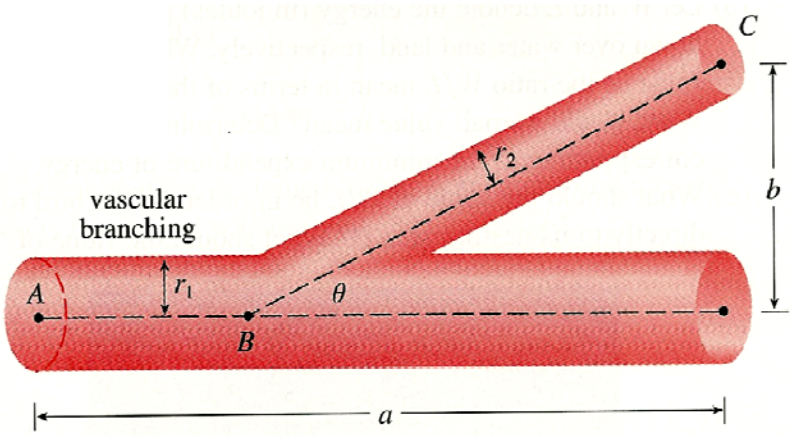


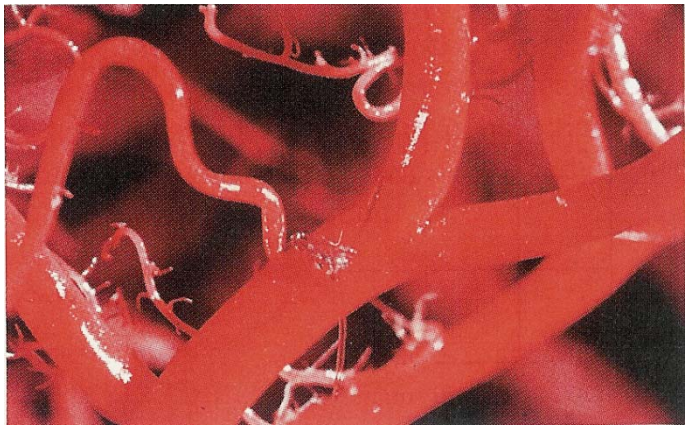
$$\triangle p \pi r^2 = \rho l \pi r^2 \frac{\triangle v}{\triangle t}$$

$$\triangle v \pi r^2 = \frac{\triangle p}{\rho l} \triangle t$$

$$\triangle t \sim \frac{r^2}{\nu}$$

$$I \sim \pi \frac{\triangle p}{\mu} \frac{r_0^4}{l}$$





Resistance:

$$R = C\frac{L}{r^4}$$

Total Resistance:

$$R = C\left(\frac{a - b\cot\theta}{r_1^4} + \frac{b\csc\theta}{r_2^4}\right)$$

The Resistance is Minimized when:

$$\cos\theta = \frac{r_2^4}{r_1^4}$$

Thomas Robert Malthus

Wikipedia

(1766～1834)

English demographer and political economist

人口學家

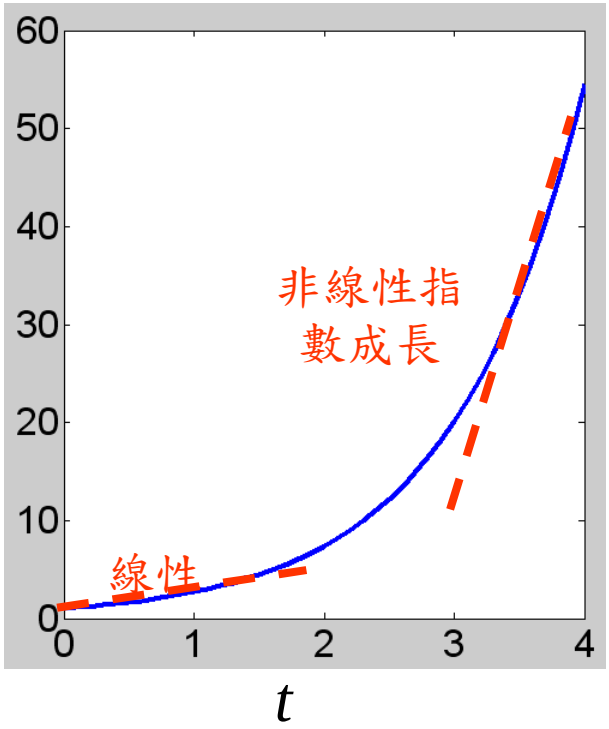
政治經濟學家

Malthusian Model

Population Growth

$$\lim_{\Delta \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

$$\frac{dp}{dt} = \alpha p \quad \Rightarrow \quad p = p_0 e^{\alpha t}$$



線性　　非線性

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

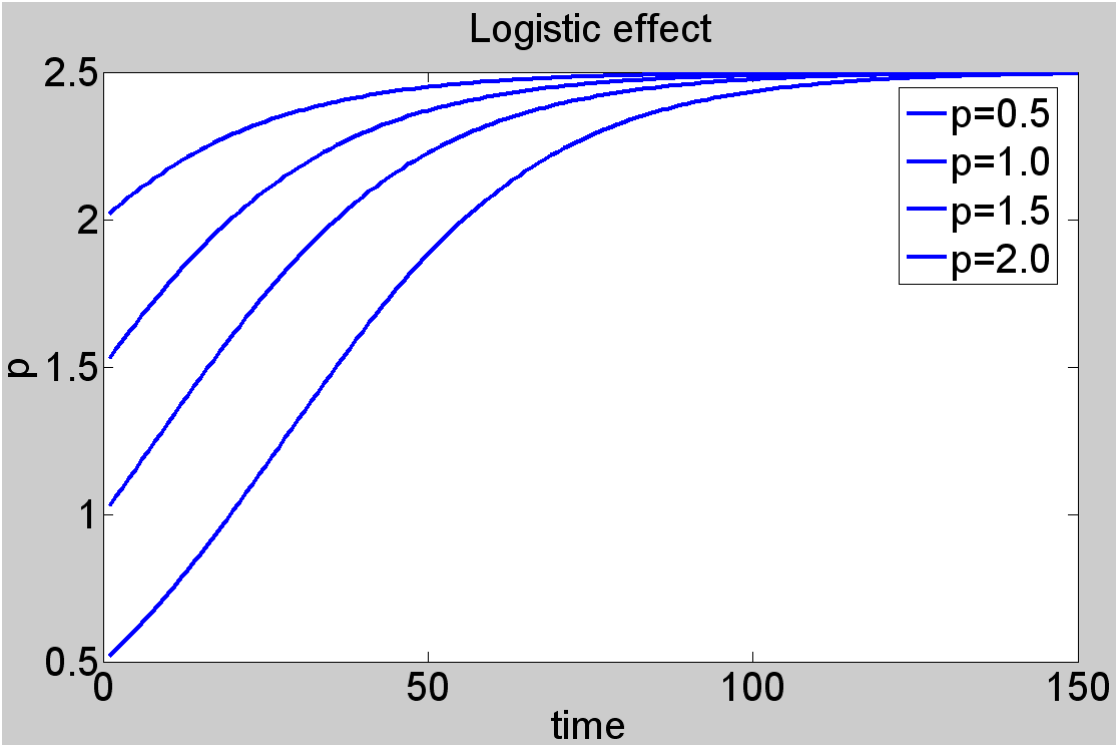
Logistic Effect

Time Series (時間序列)

$$\frac{dp}{dt} = \alpha p - \beta p^2 = \alpha p(1 - \frac{p}{\frac{\alpha}{\beta}})$$

其中 $\frac{\alpha}{\beta}$

→ 環境之承載效應



Lotka-Volterra Model

A Predator-Prey Model

Humberto D'Ancona
1926

World War One 掠食者比例變大

	Port	1914	1915	1916	1917	1918	1919	1920	1921	1922	1923
Predator	**Fiume**	12%	21%	22%	21%	36%	27%	16%	16%	15%	11%
Prey	**Trieste**	14%	7%	16%	15%	-	18%	15%	13%	11%	10%

$$\frac{dx}{dt} = x - xy$$

$$\frac{dy}{dt} = -y + xy$$

Vito Volterra

Wikipedia

(1860～1940)

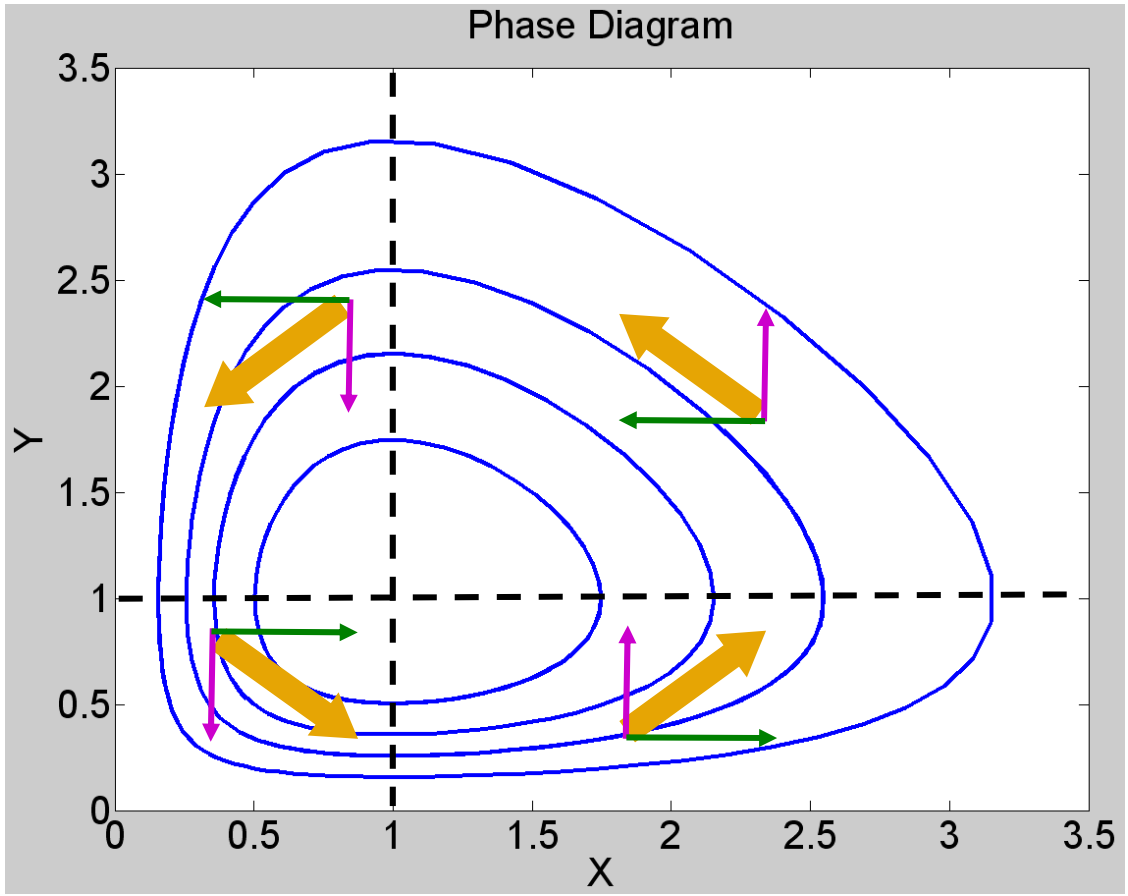
Italian
Mathematician & Physicist

Contribution:

1) Mathematical biology

2) Volterra-Lotka equations

動力系統



dx/dt = x(1-y)

$$\frac{dx}{dt} = x - xy$$

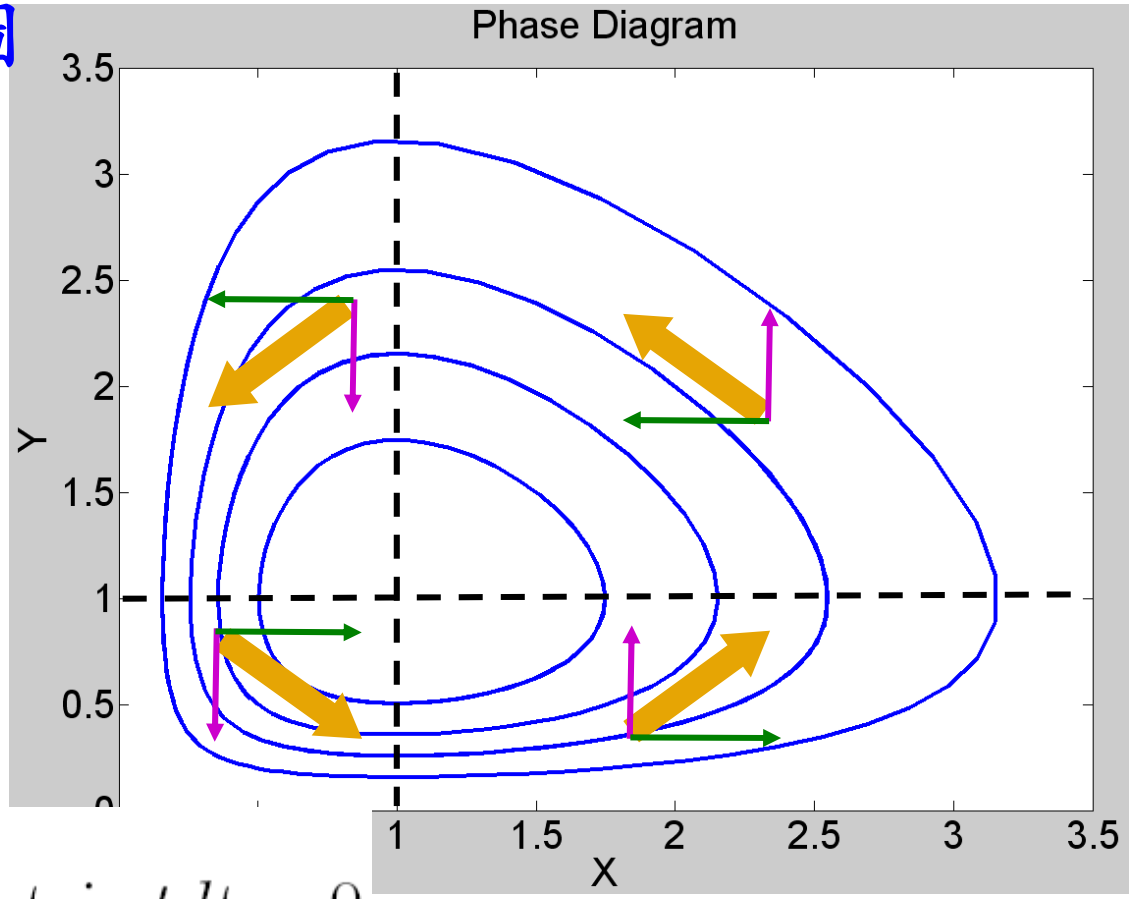
$$\frac{dy}{dt} = -y + xy$$

dy/dt = y (x-1)

相位圖

愛情動力系統

相位圖

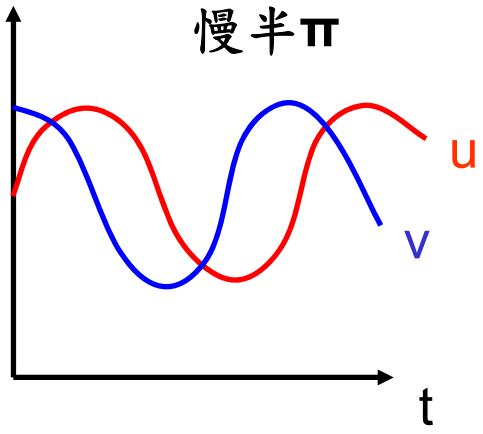


dx/dt = x(1-y)

$$\frac{dx}{dt} = x - xy$$

$$\frac{dy}{dt} = -y + xy$$

dy/dt = y (x-1)



$$\int_0^{2\pi} \cos t \sin t dt = 0$$

$u \quad v$

Cos 和 Sin 零相關、不來電！

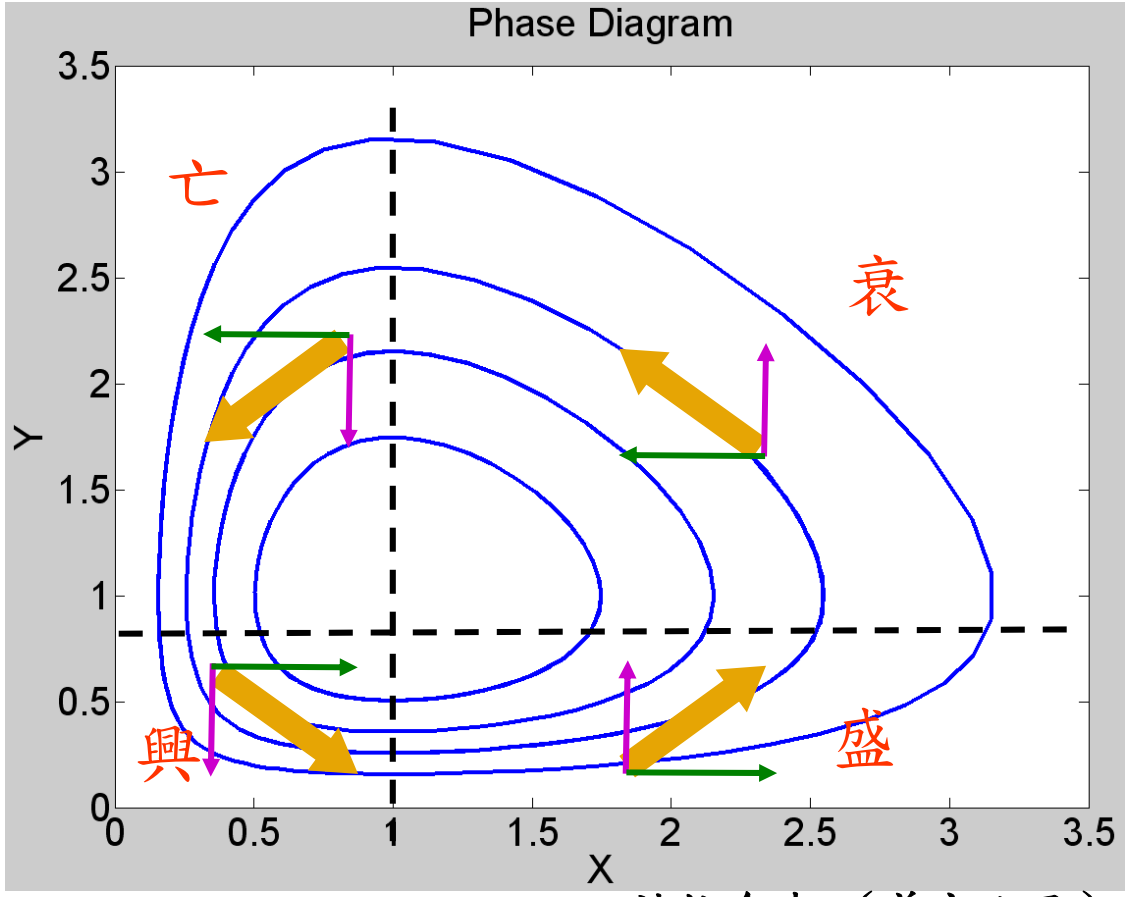
動力系統

鴛鴦蝴蝶夢，一段愛情故事；

千古興盛衰亡。

$$\frac{dx}{dt} = x - xy$$

$$\frac{dy}{dt} = -y + xy$$



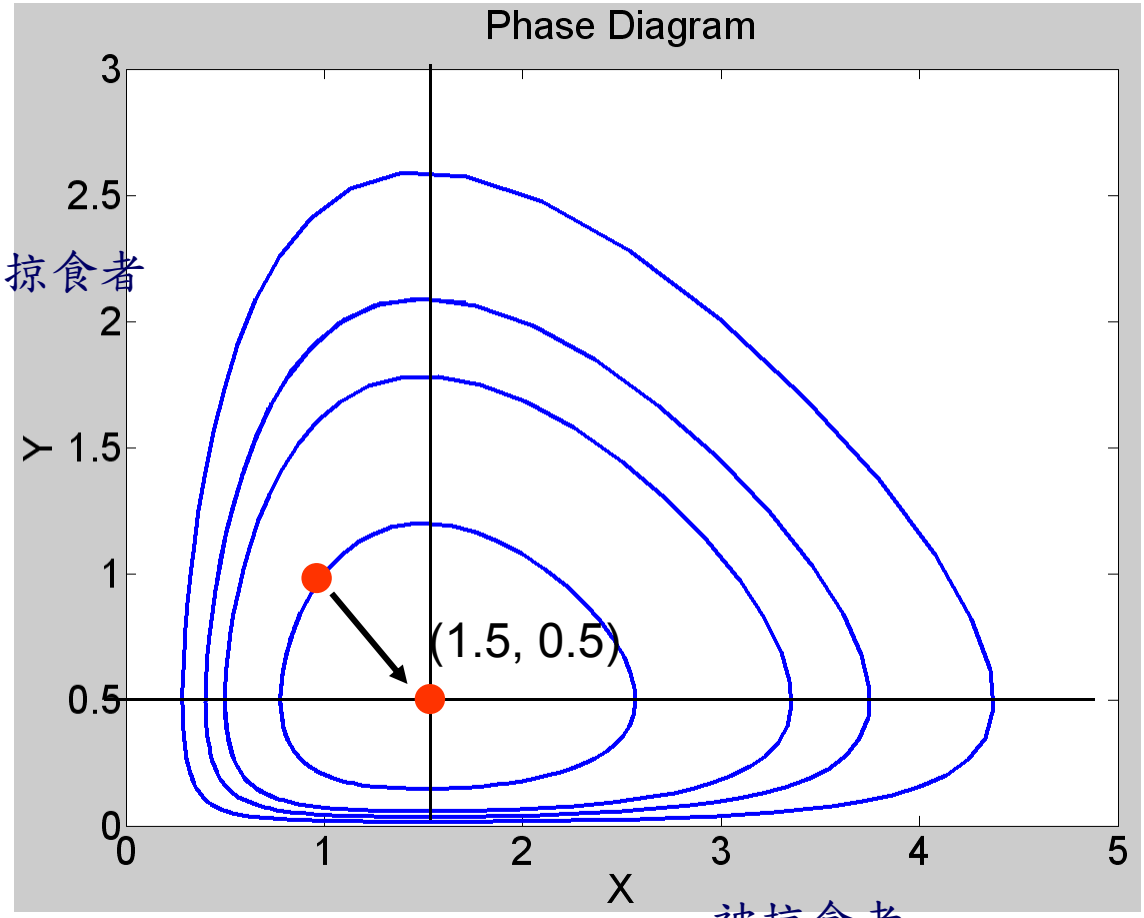
掠食者

（官僚）

被掠食者（善良人民）

$$\begin{cases} \dfrac{dx}{dt} = x - xy - hx \\ \dfrac{dy}{dt} = -y + xy - hy \end{cases}$$

$h=0.5$



Harvest effect

對掠食者與被掠食者同時進行獵取殺害，可以使生態往對被掠食者有利方向移動。

1960 DDT 事件

無間道警察

水滸傳哲學

$$\begin{cases} \dfrac{dp}{dt} = K_1 p(1-\dfrac{p}{c_1}) - \alpha_1 pq \\ \dfrac{dq}{dt} = K_2 q(1-\dfrac{q}{c_2}) - \alpha_2 pq \end{cases}$$

原方程式變數：

K_1 c_1 α_1

K_2 c_2 α_2

令

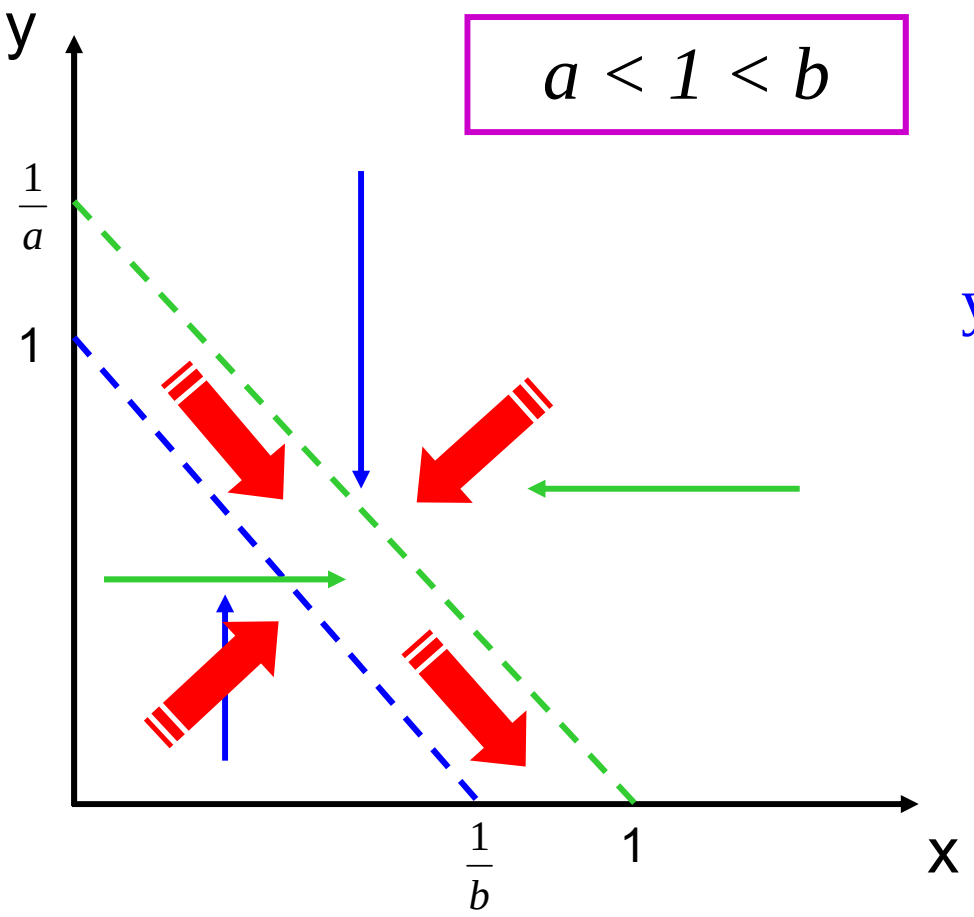
$x = \dfrac{p}{c_1}$ $\quad \dfrac{\alpha_1 c_2}{K_1} = a$

$y = \dfrac{q}{c_2}$ $\quad \dfrac{\alpha_2 c_1}{K_2} = b$

$$\begin{cases} \dfrac{dx}{dt} = K_1 x(1-x-ay) \\ \dfrac{dy}{dt} = K_2 y(1-y-bx) \end{cases}$$

強弱參數＝攻擊力/修護力， 攻擊力＝戰鬥力＊族群數目

$$\begin{cases} \dfrac{dx}{dt} = K_1 x(1 - x - ay) \\ \dfrac{dy}{dt} = K_2 y(1 - y - bx) \end{cases}$$

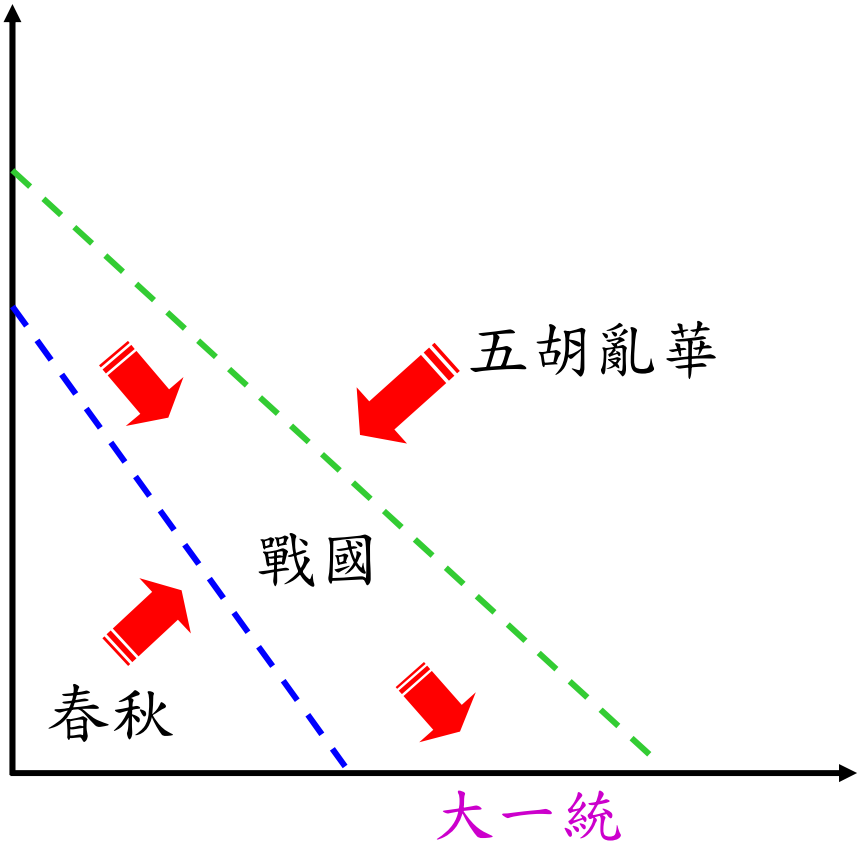


$a < 1 < b$

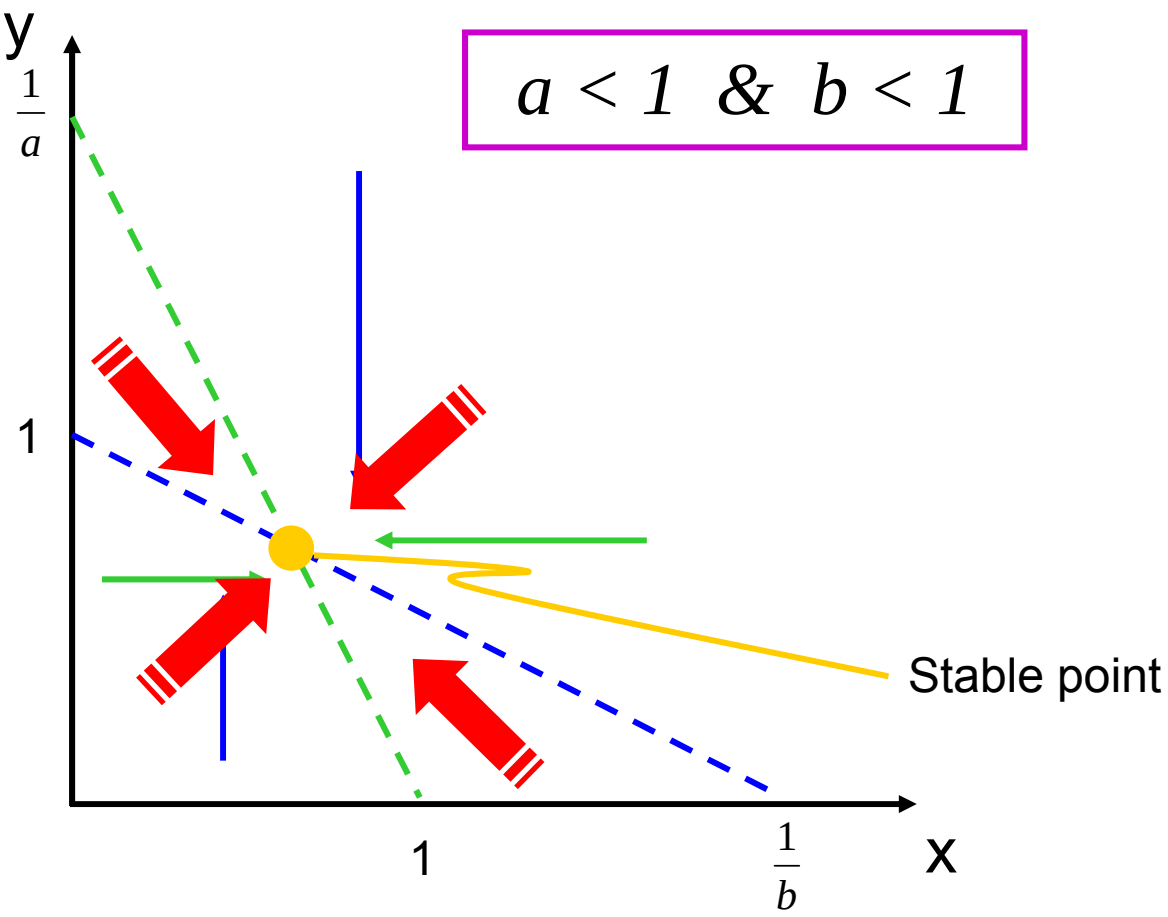
X 比 Y強

y 滅絕

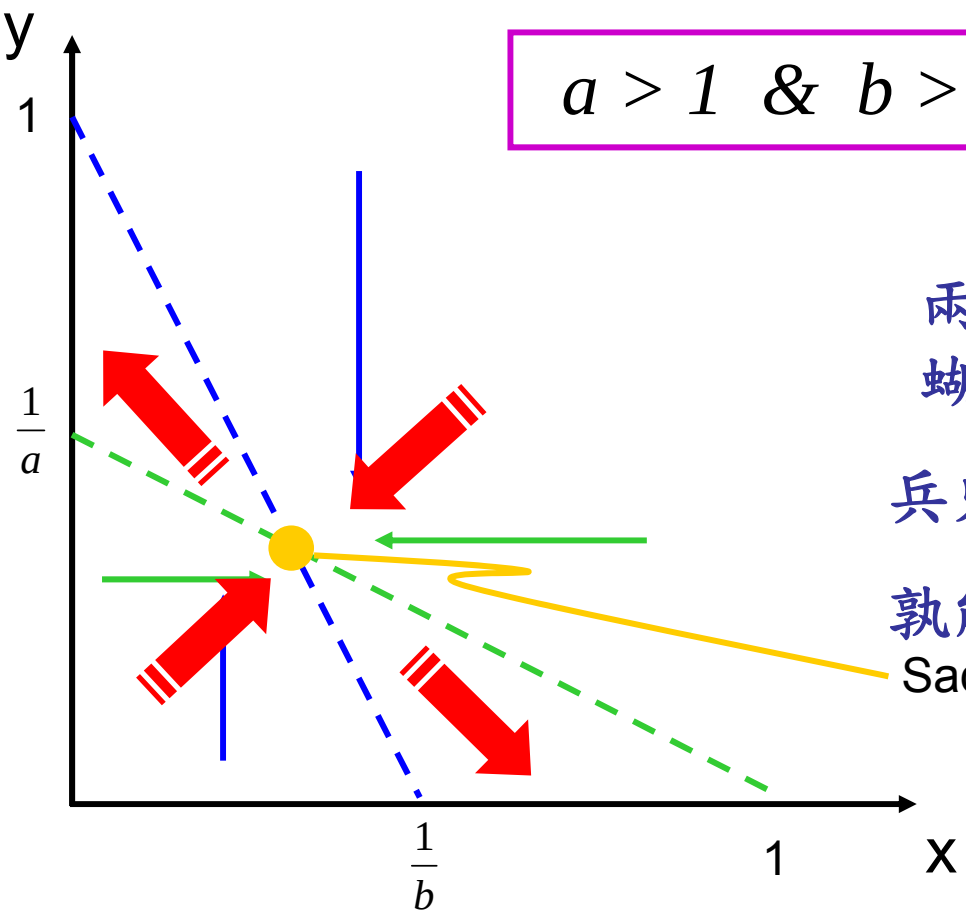
強弱之爭



$$\begin{cases} \dfrac{dx}{dt} = K_1 x(1 - x - ay) \\ \dfrac{dy}{dt} = K_2 y(1 - y - bx) \end{cases}$$



$$\begin{cases} \dfrac{dx}{dt} = K_1 x(1 - x - ay) \\ \dfrac{dy}{dt} = K_2 y(1 - y - bx) \end{cases}$$



$a > 1 \;\&\; b > 1$

兩強相爭
蝴蝶效應

兵兇戰危！

孰能必勝？

Saddle point

Boosting immunity by antiviral drug therapy: A simple relationship among timing, efficacy, and success

Natalia L. Komarova, Eleanor Barnes, Paul Klenerman, and Dominik Wodarz

演講者：李坤珀 生科四

指導老師：郭鴻基教授

2007/03/06

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Boost immunity with drug therapy

- Drug efficiency: stronger efficiency is better?
- Treating duration: longer is better?

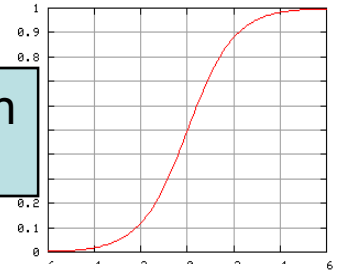
- Simulate the treatment with mathematical model

Virus suppress immunenity

- Some virus, such as HIV, HBV, and HCV, have ability to suppress immune response
- Human Immunodeficiency Virus (HIV, 人體免疫缺陷病毒)
  - About 40 million people who have been HIV infected
- Hepatitis B Virus (HBV, B型肝炎病毒)
  - 2 billion people who have been HBV infected
- Hepatitis C Virus (HCV, C型肝炎病毒)
  - 1.8 millon people who have been HCV infected

The mod

- Assumption:
  - Virus growth is density dependent
  - Immunity is saturated growth
  - Virus has ability to suppress immunity



Logistic growth

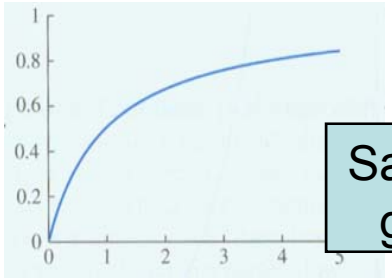
$$\dot{y} = ry\left(1 - \frac{y}{k}\right) - ay - pyz,$$

viral replication rate

Killed by immune cells

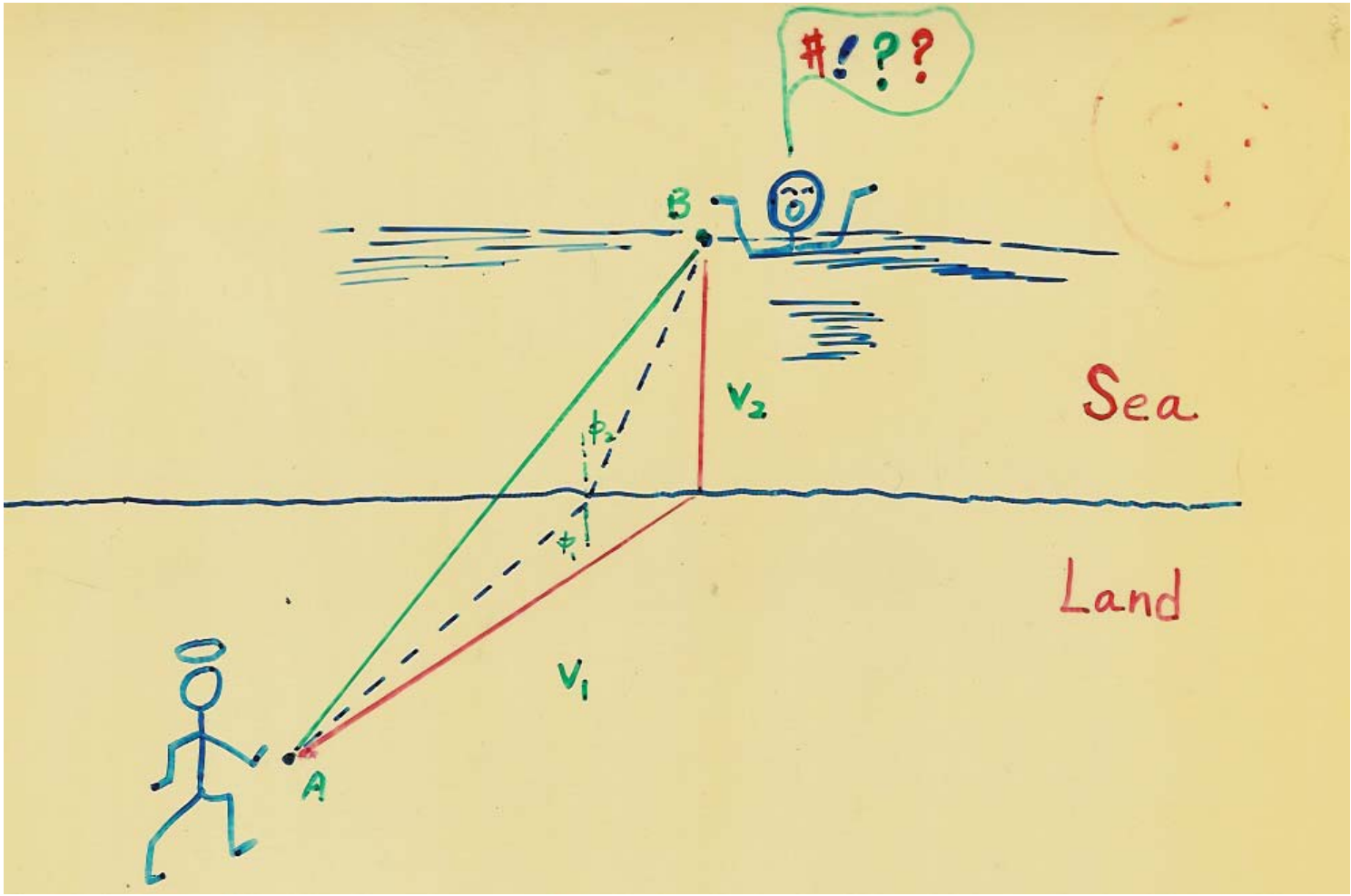
Natural degradation

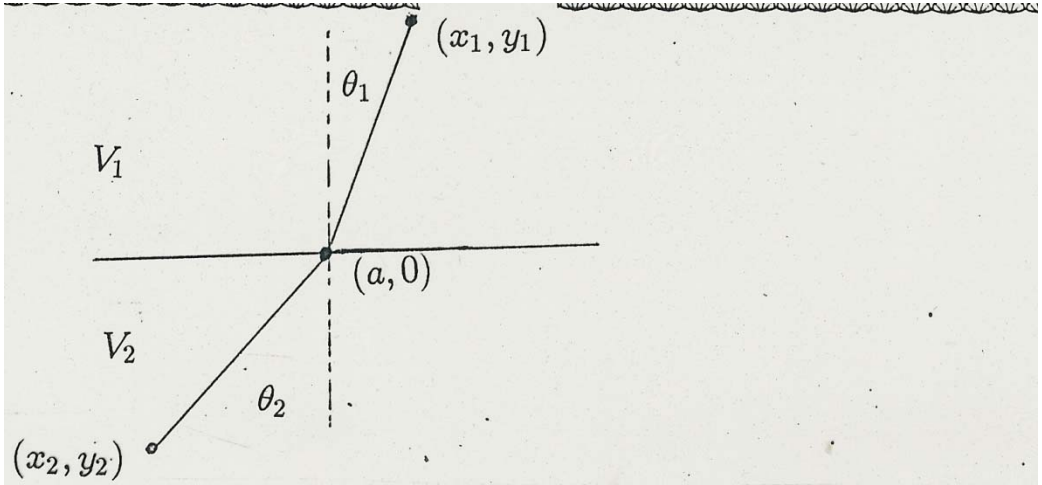
$$\dot{z} = \frac{cyz}{1 + \varepsilon y} - qyz - bz,$$



Saturated growth

Immunity suppressed by virus


#!??
B
Sea
V_2
t_2
t_1
Land
V_1
A



$$T(a) = \frac{[(x_1 - a)^2 + y_1^2]^{\frac{1}{2}}}{V_1} + \frac{[(x_2 - a)^2 + y_2^2]^{\frac{1}{2}}}{V_2}$$

$$\frac{\partial T}{\partial a} = 0$$

$$\frac{-(x_1 - a)}{[(x_1 - a)^2 + y_1^2]^{\frac{1}{2}}} \frac{1}{V_1} + \frac{(a - x_2)}{[(x_2 - a)^2 + y_2^2]^{\frac{1}{2}}} \frac{1}{V_2} = 0$$

$$\boxed{\frac{\sin\theta_1}{\sin\theta_2} = \frac{V_1}{V_2}}$$

This is the **Snell's Law.**

Hamilton's principle

Fermat's principle

Minimization

一樣觀魚多樣情！

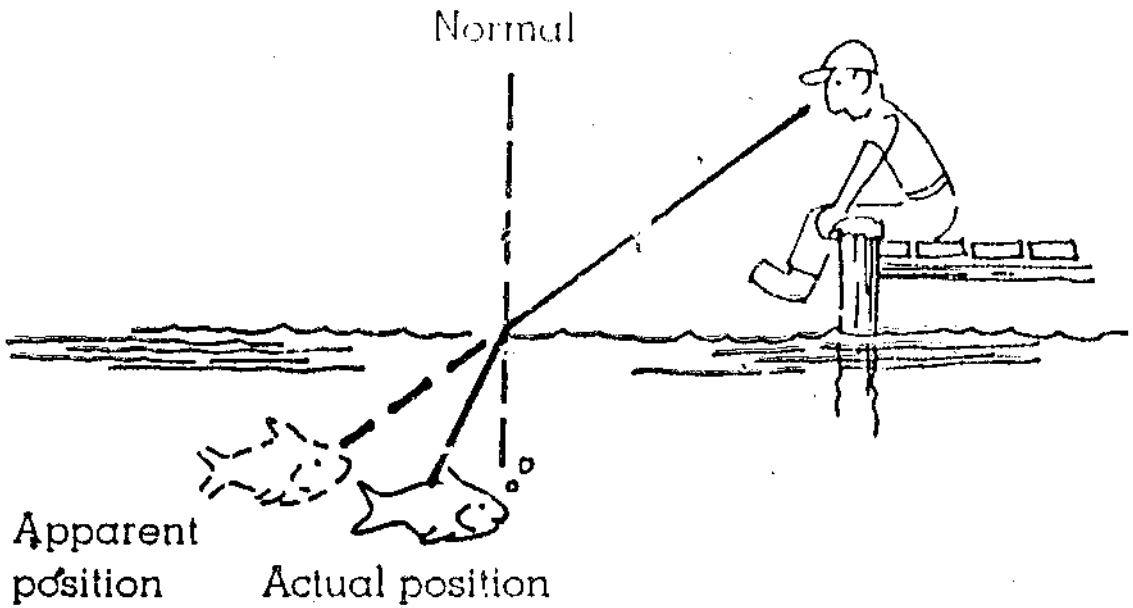


FIGURE 5.13 The refraction of light as it passes from the water into the less-dense air causes a fish to appear closer to the surface than it actually is.

（1）魚快樂嗎？
（2）熱血沸騰，立志革命！
（3）折射定律，最小原理。

Leonardo da Vinci (1452-1519)

"*Observe the movement of the surface of the water which resembles that of hair which has two motions, of which one depends on the weight of the hair and the other on the direction of the curls. Thus water forms eddying whirlpools of which one part depends on the predominant current and the other on the incidental motion and the return flow.*"

Multiple Scale Interactions
Steady and Turbulent Flows
多重尺度交互作用

$$\frac{d\mathbf{v}}{dt} + A\mathbf{v} = \mathbf{f}$$

$$\frac{dv}{dt} + \lambda v = f_0 e^{i\Omega t}$$

$$t = [\frac{1}{\Omega}]t^*$$

$$v = [\frac{f_0}{\lambda}]v^*$$

$$\frac{\Omega}{\lambda}\frac{dv^*}{dt^*} + v^* = e^{it^*}$$

$$\epsilon\frac{dv^*}{dt^*} + v^* = e^{it^*}$$

$$\epsilon = \frac{\frac{1}{\lambda}}{\frac{1}{\Omega}}$$

$$\epsilon\frac{d\mathbf{v}^*}{dt^*} + A\mathbf{v}^* = \mathbf{f}^*$$

Multiple time scales

Fast time/Slow time

Quasi-equilibrium Dynamics

Fast transient

Boundary layer Dynamics

$$\frac{D\mathbf{V}}{Dt} + 2\boldsymbol{\Omega} \times \mathbf{V} = -\frac{1}{\rho}\nabla_{\mathbf{z}}\mathbf{p} + \nu\nabla^{\mathbf{2}}\mathbf{V}.$$

$$\frac{D\mathbf{V}}{Dt} + f\mathbf{k} \times \mathbf{V} = -\nabla_{\mathbf{p}}\phi + \nu\nabla^{\mathbf{2}}\mathbf{V}.$$

Geostrophy
Rotation Dynamics

$$\epsilon\frac{D\mathbf{V}^*}{Dt^*} + \mathbf{k} \times \mathbf{V}^* = -\nabla^*_{\mathbf{p}}\phi^* + \frac{\epsilon}{\mathbf{Re}}\nabla^{*\mathbf{2}}\mathbf{V}^*.$$

Singular Perturbation Problems
Quasi-balanced Dynamics

Boundary Layer Dynamics
Nearly Invicid

$$\epsilon = \frac{1/f}{L/U}$$

Rotation time scale / Advective time scale

$$Re = \frac{L^2/\nu}{L/U}$$

Diffusion time scale / Advective time scale

A coffee lover's dream:

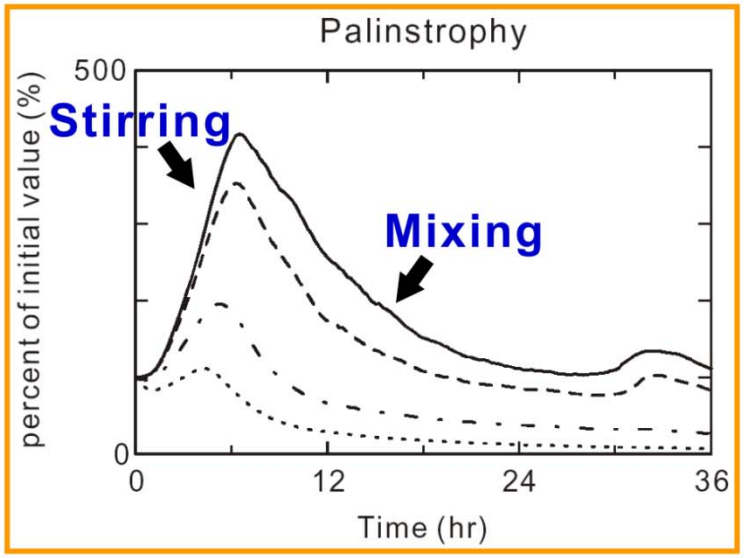
The best part of waking up, is the vortex in your cup!

$$\frac{D\theta}{Dt} = \frac{\partial\theta}{\partial t} + \vec{V}\cdot\nabla\theta = \nu\nabla^2\theta$$

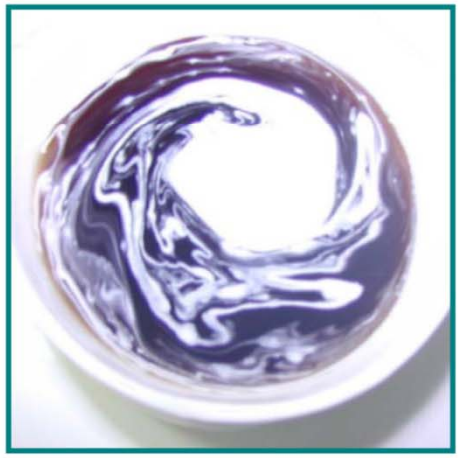
$$C = \frac{1}{2}\int \nabla\theta\cdot\nabla\theta \, dV$$

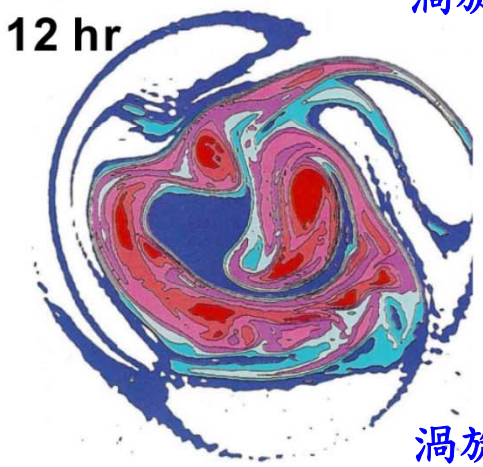
$$\frac{dC}{dt} = \int (\vec{V}\cdot\nabla\theta)\nabla^2\theta \, dV - \nu\int (\nabla^2\theta)\, dV$$

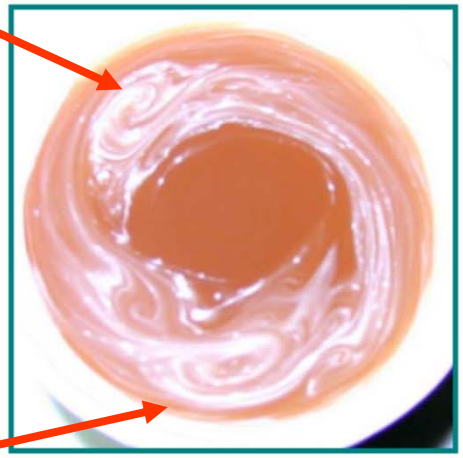
Stirring **Mixing**

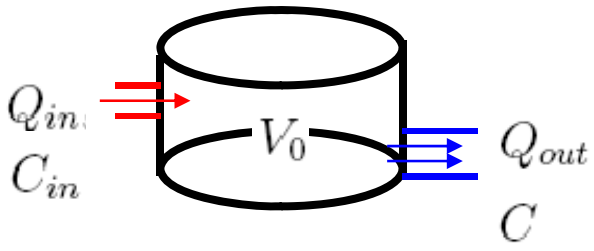


Coffee with white









$$\frac{dV}{dt} = Q_{in} - Q_{out} = -\triangle Q,$$

$$\frac{dVC}{dt} = Q_{in}C_{in} - Q_{out}C,$$

$$C(0) = 0, \quad V(0) = V_0.$$

$$V(t) = V_0 - \triangle Q t.$$

$$V\frac{dC}{dt} = Q_{in}(C_{in} - C),$$

$$\frac{dC}{dt} = \frac{Q_{in}(C_{in} - C)}{V_0 - \triangle Q t}.$$

$$C = C_{in}\left(1 - (1 - \frac{\triangle Q}{V_0}t)^{Q_{in}/\triangle Q}\right)$$

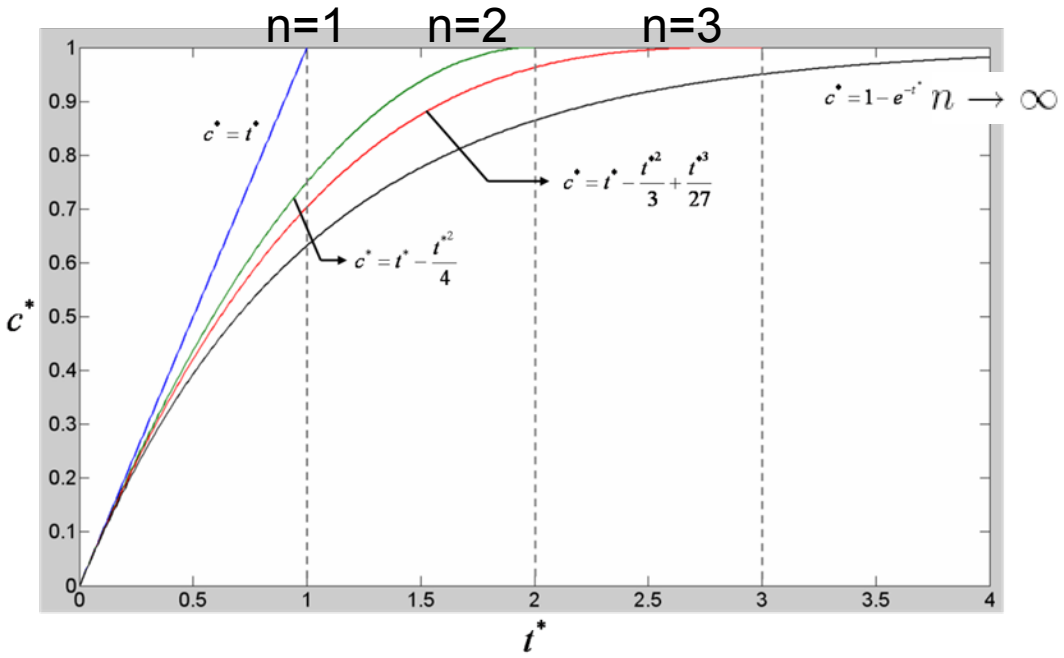
$$t = \frac{V_0}{Q_{in}}t^*, \quad C = C_{in}C^* \quad n = \frac{Q_{in}}{\triangle Q}$$

$$\frac{dC^*}{dt^*} = \frac{1 - C^*}{1 - t^*/n}$$

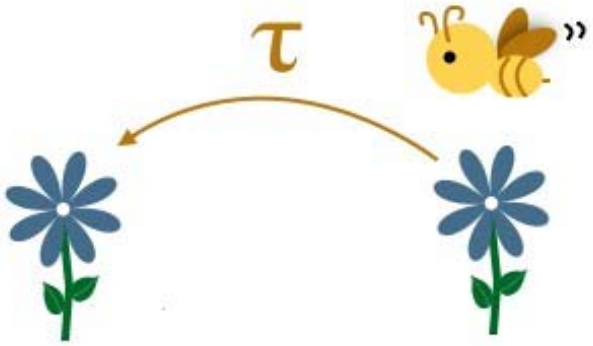
$$\triangle Q = 0, \; n \to \infty$$

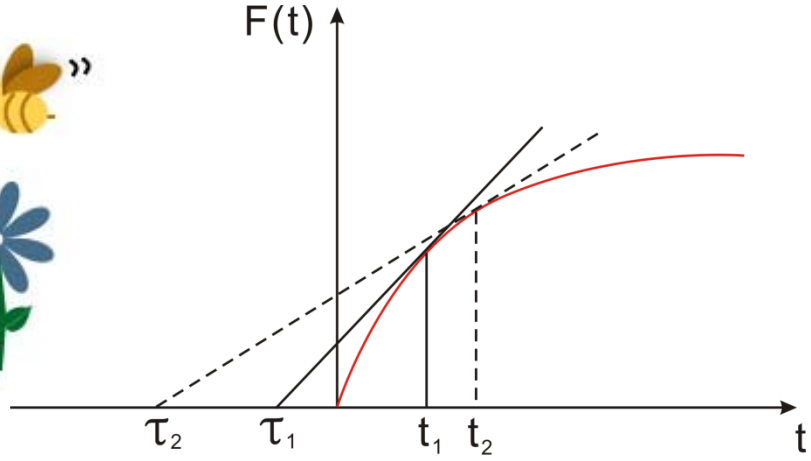
$$C^* = 1 - (1 - \frac{t^*}{n})^n.$$

$$C^* = 1 - \exp(-t^*)$$



術 方法





一朵花停留的時間：t

蜜峰從第一朵花到下一朵花的飛行時間：τ

$$g(t) = \frac{f(t)}{t+\tau} \qquad \text{nectar collection rate} = \frac{\text{food per visit}}{\text{time}}$$

$g'(t) = 0$ 有最大採集效率

$$\Rightarrow \frac{f'(t)}{t+\tau} - \frac{f(t)}{(t+\tau)^2} = 0 \quad \Rightarrow f'(t) = \frac{f}{t+\tau}$$

- James D. Murray
- Dr. John Gottman, Clinical Psychologist
- Drs. Julian Cook,
- Kristin Swanson,
- Rebecca Tyson,
- Jane White

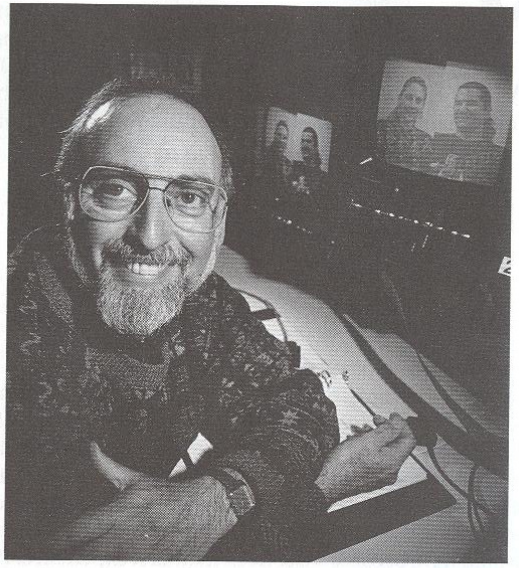


Figure 10.1. Professor John Gottman in his "Love Lab" at the University of Washington. (Courtesy of [illegible] and [illegible] Information.)

The Mathematics of Marriage, MIT Press 2002

- First Application of Mathematical Modeling in Social-Science on this Seminar.
- Problem: The divorce rate for second marriages is even higher! Why? Don't they become wiser from their first experience?
- Based on mismatches in the couples personality or modes of communication seems NOT be too successful.

- “Men are from Mars, women are from Venus”— a lack of understanding of gender differences in communication styles is at the root of marital problems.

- Another way approaching.

- Five types of married couples: Volatiles, Validators, Avoiders, Hostiles, and Hostile-detacheds. (Identified by Gottman 1994)

1. Volatiles
2. Validators
3. Avoiders
4. Hostiles
5. Hostile-detacheds

] produce stable marriages

] produce unstable (high-risk) marriages

- Characteristics (stable and unstable types):

Volatile (S) – romantic, passionate, have heated arguments with cycles of fights and sex.

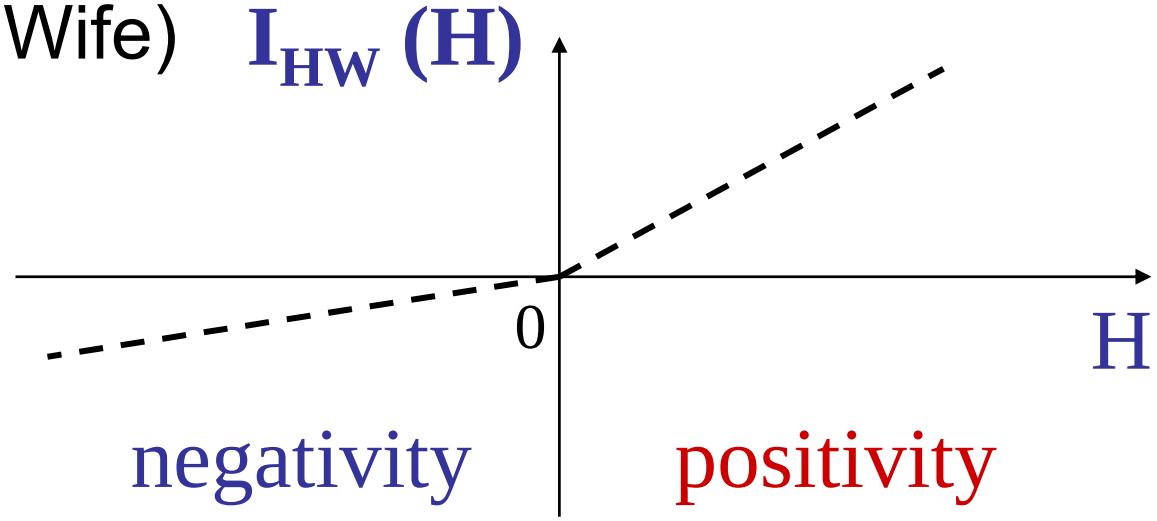
Validating (S) – calmer, intimate, value companionate marriage, shared experience rather than individuality.

Avoiders (S) – avoid confrontation and conflict, interact only in positive range of their emotions.

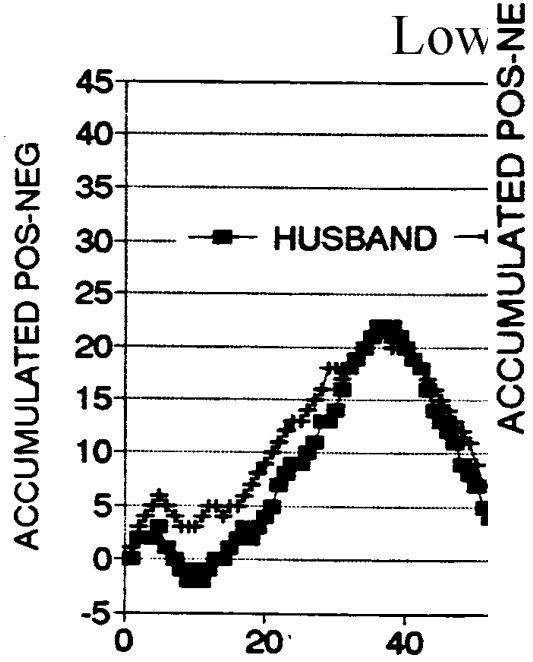
Hostile (U) – (mixed) conflict-avoiding wife, validating husband.

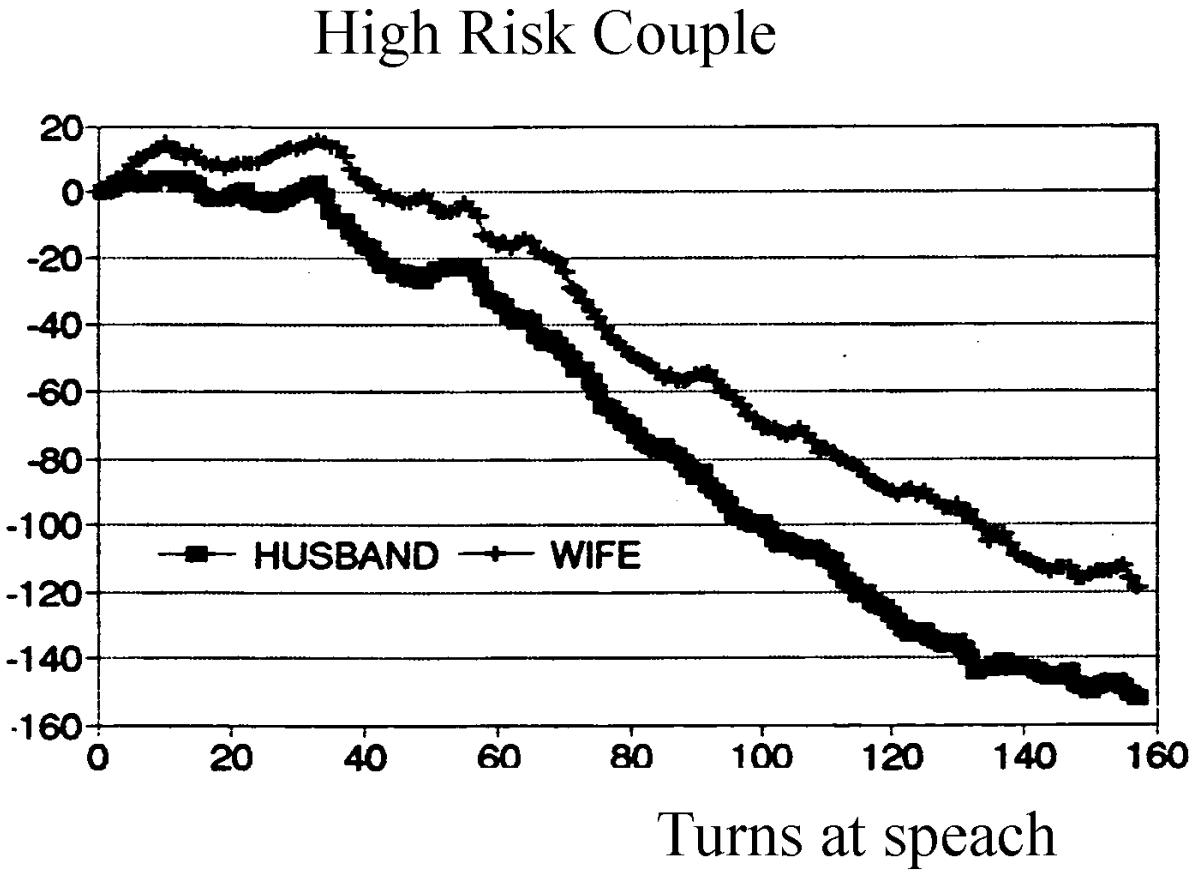
Hostile-Detached (U) – (mixed) volatile wife, validating husband

- **Five marriage types** – different interaction styles – related to interaction functions (I) in the model
- Basic model fits empirical data to a two-slope linear interaction function I (one each for the Husband and Wife)



- Observatio


Low
ACCUMULATED POS-NEG
HUSBAND
45
40
35
30
25
20
15
10
5
0
-5
0
20
40


High Risk Couple
ACCUMULATED POS-NEG
HUSBAND
WIFE
20
0
-20
-40
-60
-80
-100
-120
-140
-160
0
20
40
60
80
100
120
140
160
Turns at speach

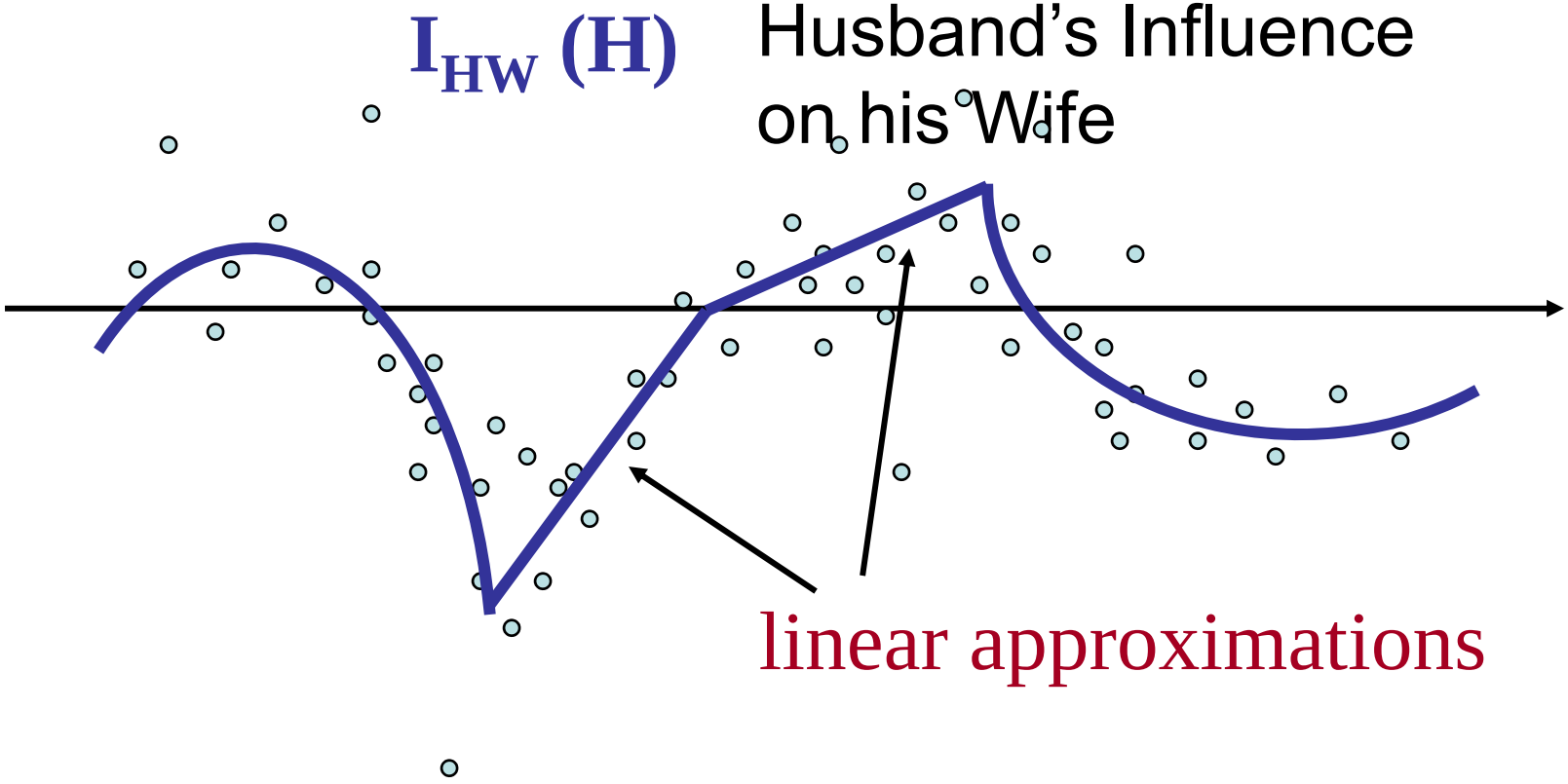
H unhappy
W happy

H happy
W happy

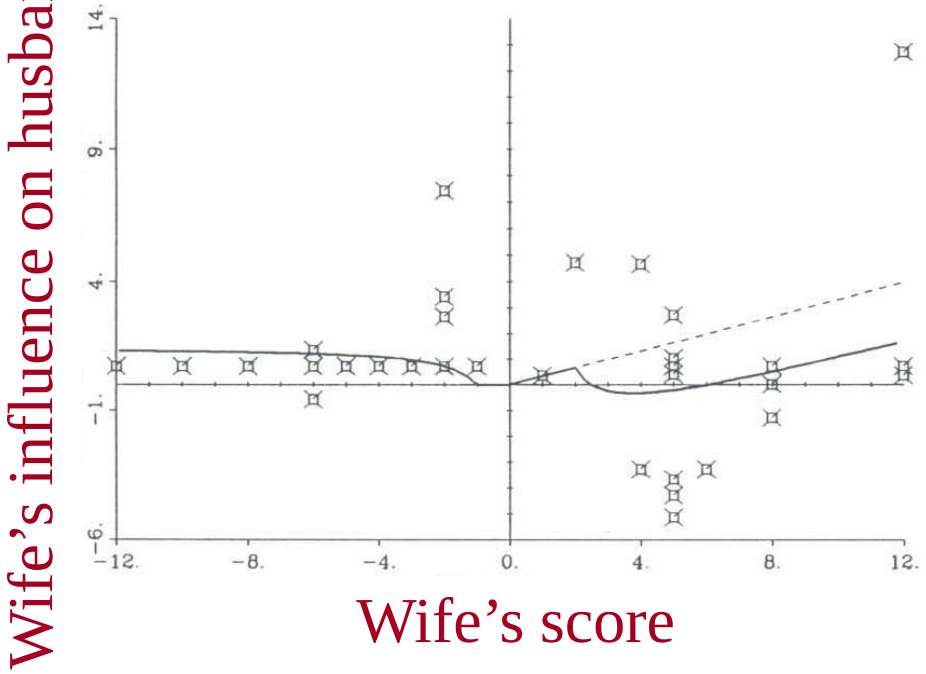
H unhappy
W unhappy

H happy
W unhappy

- **Influence Function Calculated over Complete 15 minute Discussion – Mathematical Best Fit**



Wife's influence function



Husband's influence function

- **Mathematical Model of the interaction**

  Wt , Ht represent **wife and husband's** scores at time t (turn of speech) in the discussion

Wife's score at time t + 1 = **Constant** + **Wife's previous score** + **Husband's influence on Wife**

$$W_{t+1} = a + r_1 W_t + I_{HW}(H_t)$$

Husband's score at time t + 1 = **Constant** + **Husband's previous score** + **Wife's influence on Husband**

$$H_{t+1} = b + r_2 H_t + I_{WH}(W_t)$$

- Self-Interaction (uninfluenced steady state)

$$\frac{dx}{dt} = r_1(x_0 - x), \qquad u(t) = u(0)e^{-r_1 t}$$

$$\frac{dy}{dt} = r_2(y_0 - y). \qquad v(t) = v(0)e^{-r_2 t}$$

- Marital Interactions (influenced function)

$$\frac{dx}{dt} = r_1(x_0 - x) + I_1(y), \qquad I_i(z) = \begin{cases} a_i z & if\ z > 0 \\ b_i z & if\ z < 0 \end{cases}$$

$$\frac{dy}{dt} = r_2(y_0 - y) + I_2(x).$$

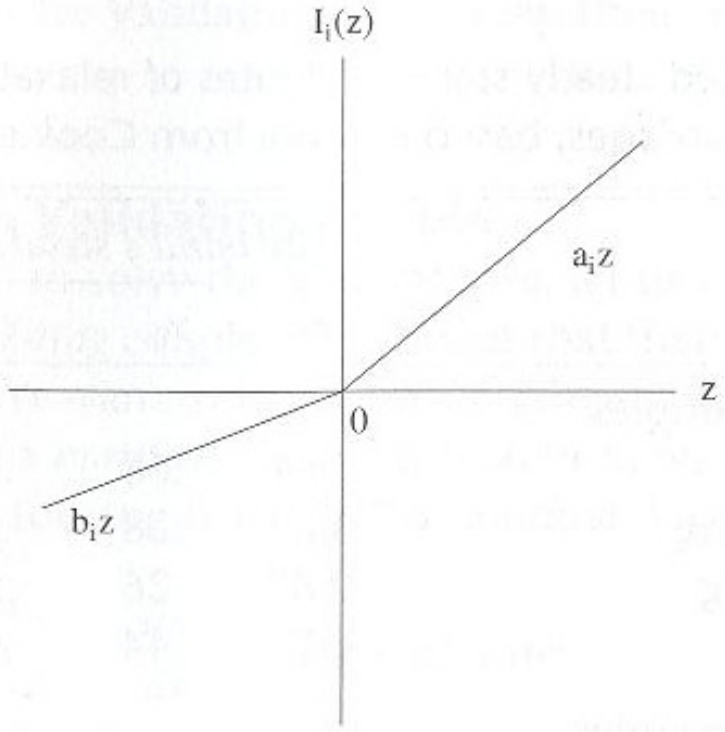


Figure 10.4. The theoretical influence function.

TABLE 10.1
Uninfluenced steady states and rates of relaxation for each spouse in five types of marriages, based on data from Cook et al. (1995)

	Husband's steady state			*Wife's steady state*		
Group	r_1	x_0	x^*	r_2	y_0	y^*
Low-risk couples						
Volatile	0.67	.68	.75	0.80	.68	.61
Validating	0.63	.38	.56	0.86	.52	.59
Avoiding	0.82	.26	.53	0.75	.46	.60
Average	0.71	.44	.61	0.80	.55	.60
High-risk couples						
Hostile	0.68	.10	.03	.51	−.64	−.45
Hostile-detached	0.60	−.42	−.50	.54	−.24	−.62
Average	0.64	−.16	−.24	.53	−.44	−.54

An example of a Validating Couple

- For a validating couple, take for simplicity, we have

$$\frac{dx}{dt} = r_1(x_0 - x) + a_1 y,$$

$$\frac{dy}{dt} = r_2(y_0 - y) + a_2 x.$$

Let (x^*, y^*) denote the equilibrium solution, then

An example of a Validating Couple

$$r_1(x_0 - x^*) + a_1 y^* = 0,$$

$$r_2(y_0 - y^*) + a_2 x^* = 0.$$

The solution is

$$x^* = [x_0 + \frac{a_1}{y_1} y_0]/[1 - \frac{a_1 a_2}{r_1 r_2}],$$

$$y^* = [y_0 + \frac{a_2}{y_2} x_o]/[1 - \frac{a_1 a_2}{r_1 r_2}].$$

An example of a Validating Couple

Stable node

$$0 < (a_1 a_2)/(r_1 r_2) < 1$$

Generally,

$$a_i / r_i < 1, b_i / r_i < 1.$$

- If not, it yields to “arms race” modeling. We don’t consider this case.

An example of a Validating Couple

- (i)if $x_0 > 0, y_0 > 0$, then

$$x^* = [x_0 + \frac{a_1}{y_1} y_0] / [1 - \frac{a_1 a_2}{r_1 r_2}] >> x_0,$$

$$y^* = [y_0 + \frac{a_2}{y_2} x_o] / [1 - \frac{a_1 a_2}{r_1 r_2}] >> y_0.$$

The validating couple have a “regulated” (low-risk)marriage.

- (ii)if $x_0 > 0, y_0 > 0$, then $x^* << x_0, y^* << y_0$. (“unregulated”, or high risk)

An example of a Validating Couple

- (iii)if $x_0 > 0, y_0 < 0$ or $x_0 < 0, y_0 > 0$, it's possible to have successful marriage (under certain condition).
- Exact Solution

Why Avoiding Conflicts Is an Effective Strategy in Marriage

- Traditionally, psychologists have thought that sweeping marital problems under the rug may be detrimental to the health of a marriage.

Why Avoiding Conflicts Is an Effective Strategy in Marriage

$$\frac{dx}{dt} = r_1(x_0 - x) + a_1 I(y),$$

$$\frac{dy}{dt} = r_2(y_0 - y) + a_2 I(x).$$

- Find the equilibrium solution,

$$r_1(x_0 - x^*) + a_1 I(y^*) = 0,$$

$$r_2(y_0 - y^*) + a_2 I(x^*) = 0.$$

Why Avoiding Conflicts Is an Effective Strategy in Marriage

- (i)if $x^* > 0, y^* > 0,$, then $I(x^*) = x^*, I(y^*) = y^*$, we have

$$x^* = [x_0 + \frac{a_1}{y_1} y_0] / [1 - \frac{a_1 a_2}{r_1 r_2}] > 0,$$

$$y^* = [y_0 + \frac{a_2}{y_2} x_o] / [1 - \frac{a_1 a_2}{r_1 r_2}] > 0.$$

- (ii)if $x^* < 0, y^* < 0,$, then $I(x^*) = 0, I(y^*) = 0$, we have

$$x^* = x_0 < 0 \quad , \quad y^* = y_0 < 0$$

Why Avoiding Conflicts Is an Effective Strategy in Marriage

- If the couple are individually negative (i.e., $x_0 < 0$, $y_0 < 0$), they are no less unhappy in marriage($x^* = x_0$, $y^* = y_0$).(successful)
- If they are naturally positive (i.e., $x_0 > 0$, $y_0 > 0$), they are much happier in marriage ($x^* > x_0 > 0$, $y^* > y_0 > 0$).(successful)

Why Avoiding Conflicts Is an Effective Strategy in Marriage

- If one spouse is negative, and the other spouse is more positive, then it's possibly successful.
- Remark The terms "successful", "failed", "stable", "unstable".

General Equilibrium Solutions

- $$r_1(x_0 - x^*) + a_1 I(y^*) = 0,$$
  $$r_2(y_0 - y^*) + a_2 I(x^*) = 0.$$
- 1. First quadrant (successful marriages)

For $x^* > 0, y^* > 0,$

$$x^* = x_0 + \frac{1}{r_1} I_1(y^*) = x_0 + \frac{1}{r_1} a_1 y^*,$$

$$y^* = y_0 + \frac{1}{r_2} I_2(x^*) = y_0 + \frac{1}{r_2} a_2 x^*.$$

Solve the equation, we get

General Equilibrium Solutions

$$x^* = [x_0 + \frac{a_1}{y_1} y_0] / [1 - \frac{a_1 a_2}{r_1 r_2}] > 0,$$

$$y^* = [y_0 + \frac{a_2}{y_2} x_o] / [1 - \frac{a_1 a_2}{r_1 r_2}] > 0,$$

We need $x_0 + \frac{a_1}{r_1} y_0 > 0, y_0 + \frac{a_2}{r_2} x_0 > 0.$

- 2. Third quadrant (unsuccessful marriages)

$$x^* = [x_0 + \frac{b_1}{y_1} y_0] / [1 - \frac{b_1 b_2}{r_1 r_2}] < 0,$$

$$y^* = [y_0 + \frac{b_2}{y_2} x_o] / [1 - \frac{b_1 b_2}{r_1 r_2}] < 0.$$

General Equilibrium Solutions

provided that $x_0 + \frac{b_1}{r_1} y_0 < 0, y_0 + \frac{b_2}{r_2} x_0 < 0.$

- 3. Second quadrant (successful marriages happen under certain condition)

$$x^* < 0, y^* > 0.$$

$$x^* = [x_0 + \frac{a_1}{y_1} y_0] / [1 - \frac{a_1 b_2}{r_1 r_2}] < 0,$$

$$y^* = [y_0 + \frac{b_2}{y_2} x_o] / [1 - \frac{a_1 b_2}{r_1 r_2}] > 0.$$

General Equilibrium Solutions

provided that $x_0 + \frac{a_1}{r_1} y_0 < 0, y_0 + \frac{b_2}{r_2} x_0 > 0.$

- 4. Fourth quadrant

$x^* > 0, y^* < 0.$

$$x^* = [x_0 + \frac{b_1}{y_1} y_0] / [1 - \frac{b_1 a_2}{r_1 r_2}] > 0,$$

$$y^* = [y_0 + \frac{a_2}{y_2} x_o] / [1 - \frac{b_1 a_2}{r_1 r_2}] < 0.$$

provided that $x_0 + \frac{b_1}{r_1} y_0 > 0, y_0 + \frac{a_2}{r_2} x_0 < 0.$

It's similar to second quadrant, reverse the role of wife and husband.

General Equilibrium Solutions

- I. If $x_0 + \frac{a_1}{r_1} y_0 > 0, y_0 + \frac{a_2}{r_2} x_0 > 0$, then

$$x^* = [x_0 + \frac{a_1}{y_1} y_0] / [1 - \frac{a_1 a_2}{r_1 r_2}],$$

$$y^* = [y_0 + \frac{a_2}{y_2} x_o] / [1 - \frac{a_1 a_2}{r_1 r_2}].$$

- II. If $x_0 + \frac{a_1}{r_1} y_0 < 0, y_0 + \frac{b_2}{r_2} x_0 > 0$, then

$$x^* = [x_0 + \frac{a_1}{y_1} y_0] / [1 - \frac{a_1 b_2}{r_1 r_2}] < 0,$$

$$y^* = [y_0 + \frac{b_2}{y_2} x_o] / [1 - \frac{a_1 b_2}{r_1 r_2}] > 0.$$

General Equilibrium Solutions

- III. If $x_0 + \frac{b_1}{r_1} y_0 < 0, y_0 + \frac{b_2}{r_2} x_0 < 0$, then

$$x^* = [x_0 + \frac{b_1}{y_1} y_0] / [1 - \frac{b_1 b_2}{r_1 r_2}] < 0,$$

$$y^* = [y_0 + \frac{b_2}{y_2} x_o] / [1 - \frac{b_1 b_2}{r_1 r_2}] < 0.$$

- IV. If $x_0 + \frac{b_1}{r_1} y_0 > 0, y_0 + \frac{a_2}{r_2} x_0 < 0$, then

$$x^* = [x_0 + \frac{b_1}{y_1} y_0] / [1 - \frac{b_1 a_2}{r_1 r_2}] > 0,$$

$$y^* = [y_0 + \frac{a_2}{y_2} x_o] / [1 - \frac{b_1 a_2}{r_1 r_2}] < 0.$$

General Equilibrium Solutions

- Example1 (Volatile couple)

$$x_0 = 0.68, y_0 = 0.68, r_1 = 0.67, r_2 = 0.80, a_1 = -0.02, a_2 = 0.10$$

$$x_0 + \frac{a_1}{r_1} y_0 = 0.68 + \frac{(-0.02)}{0.67} \times 0.68 = 0.66 > 0,$$

$$y_0 + \frac{a_2}{r_2} x_0 = 0.68 + \frac{0.10}{0.80} \times 0.68 = 0.77 > 0.$$

$$x^* = 0.66/1 = 0.66, y^* = 0.77/1 = 0.77$$

- We may use such a model for solving some problems between couples, instead of consider of gender differences in communication styles.

- Divorce prediction.

- Relationship therapy.

- **What Have We Gained from the Mathematics?**
  Three practical goals for an empirical based marital therapy & rationale for the marital experiments.

- 1. Reduce emotional inertia
- 2. Make the uninfluenced steady states of wife and husband (Wu , Hu) more positive
- 3. Make the influenced steady state (W*,H*) more positive than the uninfluenced steady state (Wu , Hu)

- New language for describing marital interaction and social influence and rationale for the marital experiments
- Concept that marriages can be classified into one of 5 types of marriage depending on the couple's interaction style:
- Stable marriages have matched interaction styles. Unstable marriages have mismatched interaction styles
- Couple's interaction data suggest specific therapy