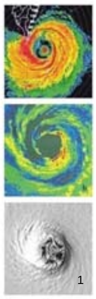
Two-Dimensional Turblence, Typhoon Dynamics , and Chebyshev Spectral Method



郭鴻基

教育部國家講座教授
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國立臺灣大學 大氣科學系

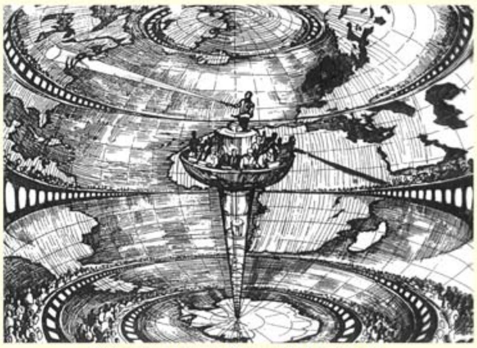
中央大學
3/24/2009

1

Politics are for the moment
An equation is for eternity

但覺高歌有鬼神
不知餓死填溝壑

Richardson's Dream

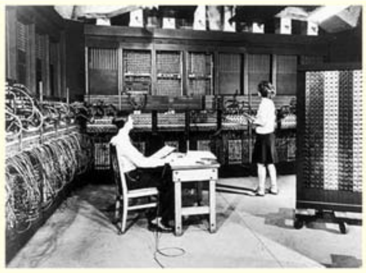


Richardson's Forecast Factory (A. Lannerback).
Dagens Nyheter, Stockholm. Reproduced from L. Bengtsson, ECMWF, 1984

64,000 Computers: The first Massively Parallel Processor

The ENIAC

Electronic Numerical Integrator and Computer



18000 vacuum tubes
70000 resistors
10000 capacitor
6000 switches

140 K Watts power

No high-level language
Assembly language

500 Flops
Function Table 0.001 s

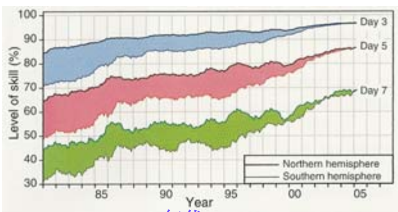
3,700,000,000 times slower than current day large computer

第一部電腦　氣象預報

4

南北半球 對於3，5，7天之預報能力隨時間的進展

預報能力指標



年代

南北差異日漸減少主要是由於近年來衛星觀測以及資料同化技術日漸成熟

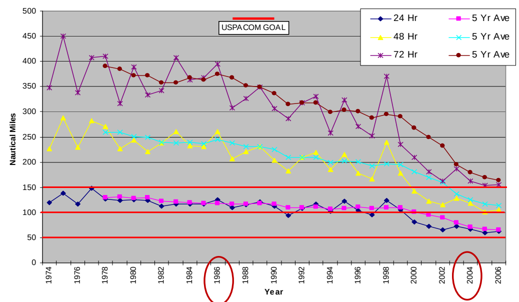
7天預報一年進步約1.5%，3天預報一年進步0.3%

5

West Pac Track Errors

Edward Fukada
JTWC

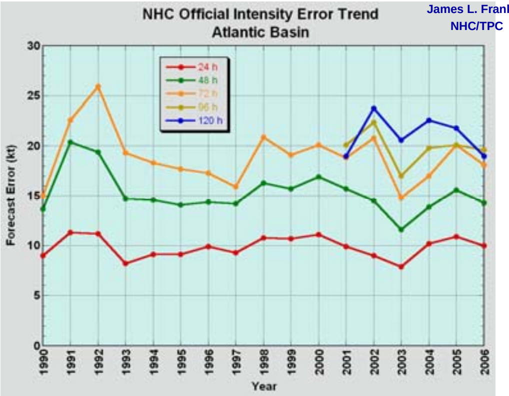
Error cut in half since 1990



美國飛機停止觀測　台灣飛機觀測

6

No progress with intensity



7

颱風潛熱與其它能量的比較

賀伯颱風的全台灣平均總雨量爲400*mm*

$400\ mm = 0.4\ m$

$0.4\ m * 1000\ kg\ m^{-3} * 2.5\times10^{6}\ J\ kg^{-1}$

$= 10^{9}\ J\ m^{2}$

$10^{9}\ J\ m^{2} * 3.5\times10^{10}\ m^{2}$

$= 3.5\times10^{19}\ J \sim 10^{20} J$

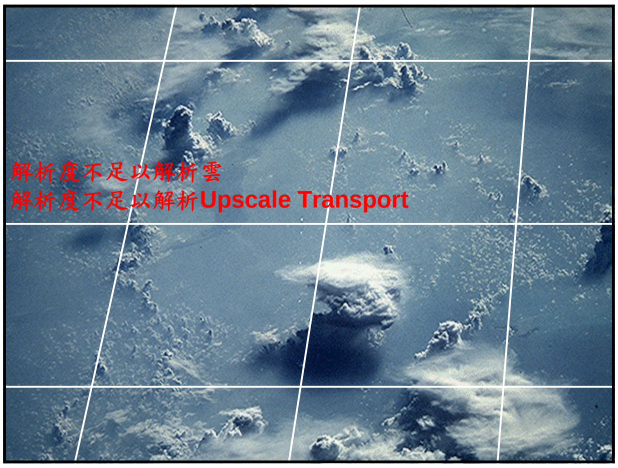
$^{1}_{0}n + ^{235}_{92}U \rightarrow ^{142}_{56}Ba + ^{91}_{36}Kr + 3^{1}_{0}n$

$1.68 * m * 10^{13} J/mol$

$\Rightarrow 1.46\times10^{6}\ kg\ U^{235}$ ($6*10^{6}$ mol)

能量估計值		備註
賀伯颱風降雨總潛熱能量	10^{20} J	可使台灣整層大氣增溫100度
台灣一年用電量	$5*10^{17}$ J	需數百年用電量才相當
全世界核子彈爆炸釋放能量	$2*10^{19}$ ~$2*10^{20}$ J	與賀伯颱風同等級
核戰後燃燒釋放能量	$2*10^{20}$ J	與賀伯颱風同等級
地球一天接受的太陽能量	$1.5*10^{22}$ J	數百個賀伯颱風
Tunguska隕石撞地球 （西元1908年，西伯利亞）	10^{16} J	賀伯颱風的萬分之一
火流星撞地球（恐龍滅絕？）	$4*10^{23}$ J	數千個賀伯颱風

8



讀 算 寫

幾何
代數
微積分
電腦計算繪圖
數學建模/科學計算
Mathematical Modeling
Scientific Computing

+ -	x /
加、減	乘、除
線性	非線性
大題大作	小題大作

10

Fovell, 2008 高雄

This model will be a simplification and an idealization, and consequently a falsification. It is to be hoped that the features retained for discussion are those of greatest importance in the present stage of knowledge.

Turning The Chemical Basis of Morphogenesis

11

“Six monkeys, set to strum unintelligently on typewriters for millions of years, would be bound in time to write all the books in the British Museum.” Huxley

君子致用在乎經邦，經邦在乎立事，立事在乎師古，師古在乎隨時。必參古今之宜，
窮終始之要，始可以度其古，中可以行於今。通典

共49個字，假設中文常用字爲1000字，共有10^147個選擇

地球歷史 10^18 sec
10^10 一百億隻猴子在打字，假設每秒鐘打一萬字 10^4，

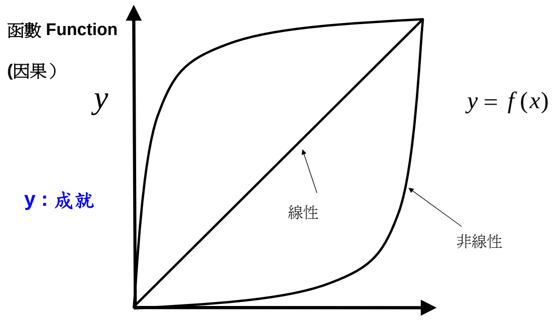
10^10*10^18*10^4 = 10^32

10^32/10^147=10^(-115) ~ 0 機率爲零 ，不可能的巧合！

研究學問是苦心孤詣的事業！ 不要人云亦云！

12

一分耕耘 ， 一分收穫？？



$$\frac{dy}{dx} > 0 \Rightarrow \frac{d^2y}{dx^2} = ?$$

x : 努力

13

你快樂嗎？一個簡單的生涯規劃動力系統

u：快樂指數
x:考試作業量
y:玩魔獸的時間

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

天縱英明的資優生 <0 >0 <0 <0 >0

考試越少越不快樂，
玩魔獸的時間越多越不快樂

$\frac{\partial u}{\partial x}$>0 考試越多越快樂
$\frac{\partial u}{\partial y}$<0 電動越玩越不快樂
人的個性

$\frac{dx}{dt}$
$\frac{dy}{dt}$
人的境遇

個性+境遇=人生
相形不如論心
論心不如則術
形不勝心
心不勝術 荀子非相

14

$$F \sim \rho v^2 l^2$$
$$P \sim \rho v^3 l^2$$
$$G \sim \frac{l^3}{n}$$
$$l^2 \sim G^{2/3} n^{2/3}$$
$$P = nA \sim \rho v^3 l^2 \sim \rho v^3 G^{2/3} n^{2/3}$$
$$v \sim n^{1/9}$$

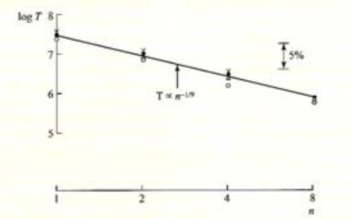


Figure 1.13. The −1/9 power-law dependence of the rowing time T on the number of oarsmen n (solid line). This may be compared with racing times over 2000 m, all at calm or near calm conditions: Δ, 1964 Olympics, Tokyo; ●, 1968 Olympics, Mexico City; ×, 1970 World Rowing Championships, Ontario; ○, 1970 Lucerne International Championships. After McMahon (1971).

Sir Isaac Newton
(1642-1727)

Isaac Newton

Principia 1687

Nature and nature's law
lay hid in night,
God said,
Let Newton be,
and all was light. **A. Pope**

16

Edmund Halley (1656–1742)

Edmund Halley was a contemporary and friend of Isaac Newton.

He was largely responsible for persuading Newton to publish his *Principia Mathematica*.

Halley and his Comet

Halley's analysis of what is now called Halley's comet is an excellent example of the scientific method in action.

If the astronomers can make
Accurate 76-year forecasts,
Why can't the Meteorologists do
the same?

Size of the problem
大氣海洋自由度無限 ＋ 熱力學

Order versus chaos
大氣海洋的混沌、蝴蝶效應

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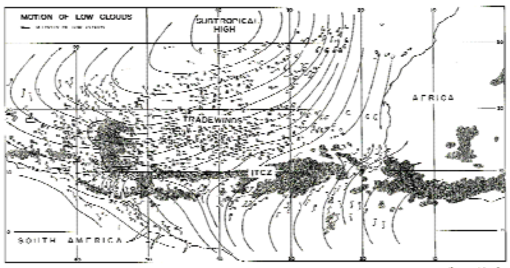
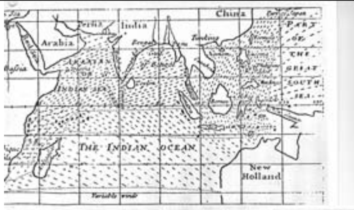
Halley 1686

First proposed the atmospheric motion is connected with the distribution of sun heat
(follow the sun in the daily scale; thus wind is westward.)

Fujita, 1971





Halley's World Map of Wind Circulations
Fig. 8.1

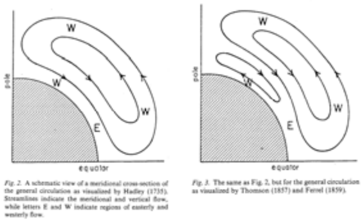
Halley 1686

Sun heat

D'Alembert 1746

Solar and Lunar force

D'Alembert's Map of the Winds in the Lower Latitudes
Fig. 8.2



Hadley (1685-1758)

Distribution of **sun** heating (north and south; seasonal scale)

Earth **rotation** (conservation of angular momentum)

Thomson (1857)
Ferrel (1859)

Centrifugal force

Coriolis 1835

D'Alembert 1746

Math. Model for Atmospheric Motion in aqua-planet
(Won the 1746 Berlin Academy's Award; Euler's endorsement)

Solar and Lunar Force

Fourier 1768-1830

Why the earth not heating up when receive sun energy continuously?

Heat emission or diffusion (by IR)

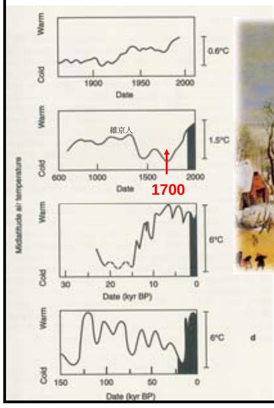
His calculations showed a very cold surface (No green house effect)

Arrhenius 1896

CO_2 green house effect, but were dismissed by scientists [WHY??]

20

氣候變遷 **Warming trend begins 1700A.D.**

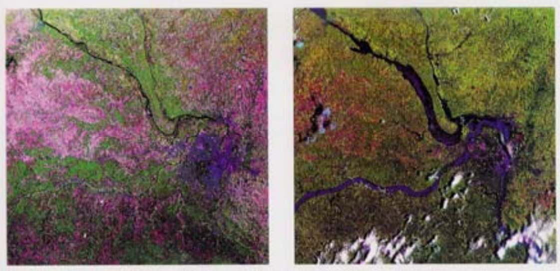


Bruegel, Pieter, the Younger
< Winter Landscape (1601) >

21

風雨之不時，是無世而不常有之。 荀子天論

1988 年際變化 季節預報 1993



Heavy rains in the summer of 1993 produced floods along most of the Mississippi River in the central United States, as shown in these Earth satellite photographs of St. Louis, Missouri on July 4, 1988 (left) and July 18, 1993 (right). Extreme climatic events may be increasing in frequency as a consequence of added radiation absorbing gases in the atmosphere.

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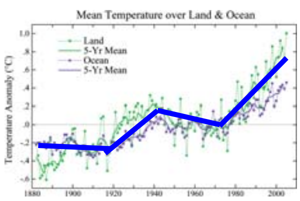
SPECIAL REPORT
Business & Media Institute
May 17, 2006

FIRE AND ICE

Journalists have warned of climate change for 100 years, but can't decide weather we face an ice age or warming



A New York Times-line

"MacMillan Reports Signs of New Ice Age"	"America in Longest Warm Spell Since 1776; Temperature Line Records a 25-Year Rise"	"Scientists Ponder Why World's Climate is Changing; A Major Cooling Widely Considered to Be Inevitable"	"Past Hot Times Hold Few Reasons to Relax About New Warming"
Sept. 18, 1924	March 27, 1933	May 21, 1975	Dec. 27, 2005
-	+	-	+

23

JANUARY 31

1977

24

Science Digest

February, 1973

Reports that the world's *climatologists are agreed* that "we must prepare for the next ice age."

Time magazine's June 24, 1974, story showed how Arctic snow and ice had grown from 1968 to 1974.

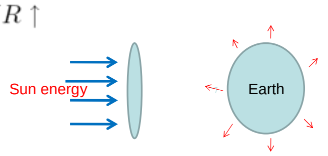
A P R I L 3

2 0 0 6

26

$$C\frac{dT}{dt} = S\downarrow - IR\uparrow$$

比熱 specific heat



$$S\downarrow = \pi a^2 \, s(1-\alpha) \qquad IR\uparrow = 4\pi a^2 \, \epsilon\sigma T^4$$

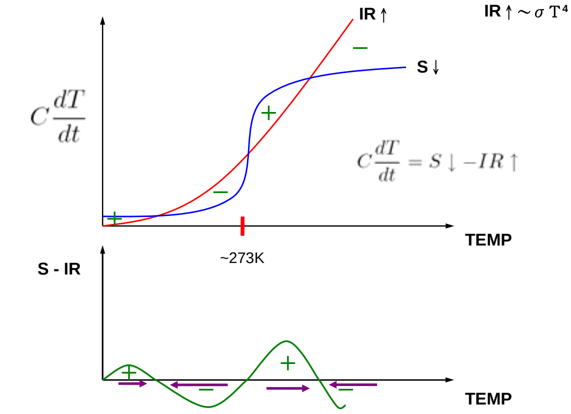
反照率 albedo

比熱　　海水　深層海水

反照率　　冰雪　　雲（IPCC沒討論的因素，氣象最大的挑戰）

太陽常數　天文因素　太陽物理

27

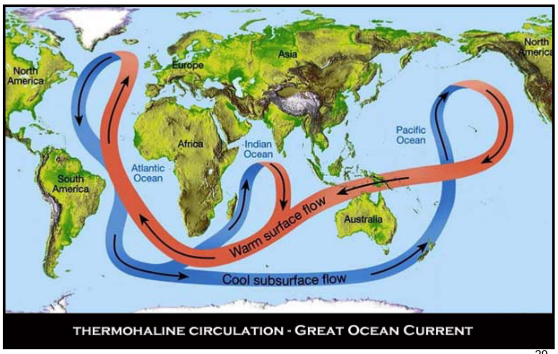


$$C\frac{dT}{dt} = S\downarrow - IR\uparrow$$

2 stable multiple equilibria

28

洋流－深海循環



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Uncertainty over weakening circulation

Petr Chylek
Los Alamos National Laboratory
Los Alamos, New Mexico

Barbara Goss Levi's Search and Discovery story (PHYSICS TODAY, April 2006, page 26) discusses evidence of weakening ocean circulation and its possible connection to global warming. The Atlantic Ocean circulation across 25° N latitude has been used as a benchmark

Bryden and Longworth *Nature* 2005

1957	$22.9 \pm 6\ SV$
2004	$14.8 \pm 6\ SV$
Net	$8.1 \pm 6\ SV$

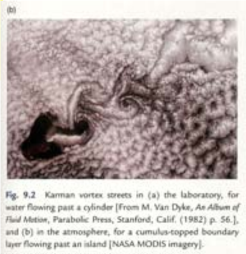
$$1\ SV = 10^6 m^3 s^{-1}$$

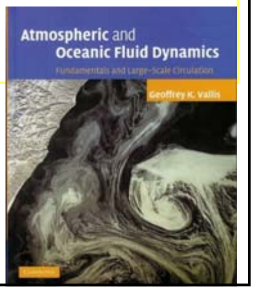
correct result. It is a mystery how such an error was missed by Levi and by the editors and reviewers of the original paper. The observed change of 8.1 Sv is well within the uncertainty of the measurement. The correct conclusion from

$$8.1 \pm 12\ SV$$

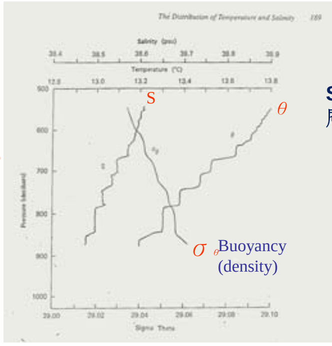
20th Century

Geophysical Fluid Dynamics (GFD)
Atmospheric Oceanic Fluid Dynamics (AOFD)
is for those interested in doing research in the physics, chemistry, and/or biology of Earth fluid environment.

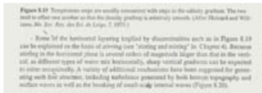




Ocean Spice



Stratification
層化



32

Multiple Scale Interactions in Vortex

Wave mean flow interaction in stable stratified fluid
Turbulent feed back to the vortex mean flow

2D turbulence

33

熱力學 ＋ 流體力學

Euler 1755

$$\frac{d}{dt}\int_{v_m} \rho\vec{v}\, dv = -\int_{\partial v_m} p\, d\vec{s}$$

$$\int_{v_m} \rho\frac{d\vec{v}}{dt}\, dv = -\int_{v_m} \nabla p\, dv$$

$$\rho\,\frac{d\vec{v}}{dt} = -\nabla p$$

Lagrange 1781

$$\frac{\partial \vec{u}}{\partial t} + \vec{\zeta}\times\vec{u} = -\frac{1}{\rho}\nabla p - \nabla K - \nabla\Phi$$

Rotation Vortex

Lorentz Force Law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$$

$$\mathbf{F} = q(-\nabla V + \mathbf{v}\times\mathbf{B})$$

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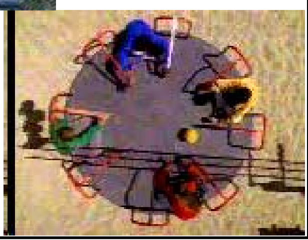
旋轉
Rotation

Coriolis Force

Non-inertial Frame



Waves with zero potential vorticity

Non-rotation rotation rotation

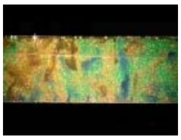
Gravity waves

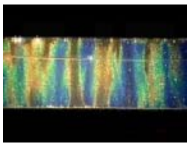
Kelvin Waves
Edge waves

36

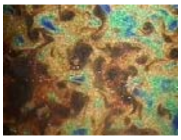
3D

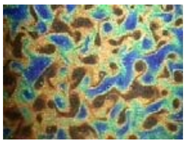
2D (strong rotation)





Taylor columns Vortex Tubes





Vortices with sharp edge

37

Kyoto Univ. GFD group

2D Turbulence

Stratification and/or Rotation
Vortex Waves Turbulence

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} = \nu \nabla^2 \zeta$$

$$u = -\frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \psi}{\partial x}.$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial(\psi, \zeta)}{\partial(x, y)} = \nu \nabla^2 \zeta$$

38

$$\frac{d}{dt}\int E(k)dk = 0. \qquad \frac{d}{dt}\left(\int k^2 E(k)dk\right) = \frac{d}{dt}\int Z(k)dk = 0$$

$$\frac{d}{dt}\left(\int (k-k_1)^2 E(k)dk\right) > 0$$

$$\frac{d}{dt}\left(\int k^2 E(k)dk + k_1^2 \int E(k)dk - 2k_1 \int kE(k)dk\right) > 0$$

$$\frac{d}{dt}\left(\frac{\int kE(k)dk}{\int E(k)dk}\right) < 0,$$

Kinetic energy moves toward large scales

$$\frac{d}{dt}\left(\int (k^2-k_1^2)^2 E(k)dk\right) > 0$$

$$\frac{d}{dt}\left(\int k^2 Z(k)dk + k_1^4 \int E(k)dk - 2k_1^2 \int k^2 E(k)dk\right) > 0$$

$$\frac{d}{dt}\left(\frac{\int k^2 Z(k)dk}{\int Z(k)dk}\right) > 0,$$

Enstrophy moves toward small scales

39

Non-divergent barotropic model (Nearly Inviscid Fluid)

$$\frac{\partial}{\partial t}\zeta + J(\psi, \zeta) = \upsilon \nabla^2 \zeta \qquad \nabla^2 \psi = \zeta$$

The energy and enstrophy relations

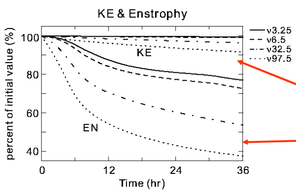
$$\frac{d\mathbf{E}}{dt} = -2\,\upsilon\, Z \qquad \mathbf{E} = \iint \frac{1}{2}\left(u^2 + v^2\right) dx\, dy \quad \text{kinetic energy}$$

$$Z = \iint \frac{1}{2}\zeta^2\, dx\, dy \quad \text{enstrophy}$$

$$\frac{dZ}{dt} = -2\,\upsilon\, \mathrm{P} \qquad \mathrm{P} = \iint \frac{1}{2}\nabla\zeta \cdot \nabla\zeta\, dx\, dy \quad \text{palinstrophy}$$

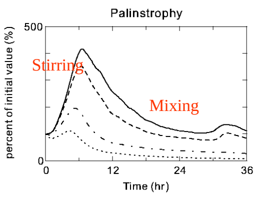
Batchelor 1969

40



KE nearly conserved

Enstrophy cascade



Selective Decay of 2D turbulence

41

$E \sim p'^2 / L^2$ (KE) geostrophy

$Z \sim p'^2 / L^4$ (Enstrophy)

KE nearly conserved $L \sim p'$

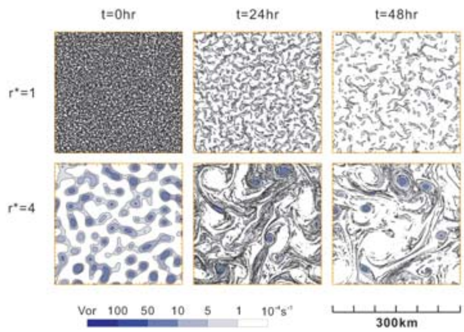
Enstrophy cascade L↑ (L increase Z decrease)

Selective Decay of 2D turbulence

The vortices become, on the average, larger, stronger, and fewer.

Merger and Axisymmetrization Dynamics

42



Fewer and stronger vortices !!!

小尺度變大尺度

Coherent structure with filamentations in 2-D turbulence

43

Weiss(1981,1991), Rozoff et al. (2004)

$$\frac{D}{Dt}(\nabla\zeta) = -J(\nabla\psi,\zeta)$$

$$\rightarrow \nabla\zeta(t) \propto \exp(\lambda t) \quad \lambda = \pm\frac{1}{2}\sqrt{Q} = \pm\frac{1}{2}\sqrt{S_1^2 + S_2^2 - \zeta^2}$$

$$S_1 = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \quad \text{(stretch deformation)}$$

$$S_2 = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \quad \text{(shear deformation)}$$

Q > 0 (strain dominates)
→vorticity gradient will be stretched
Q < 0 (vorticity dominates)
→vortex is stable (survival of eyewall meso-vortices)

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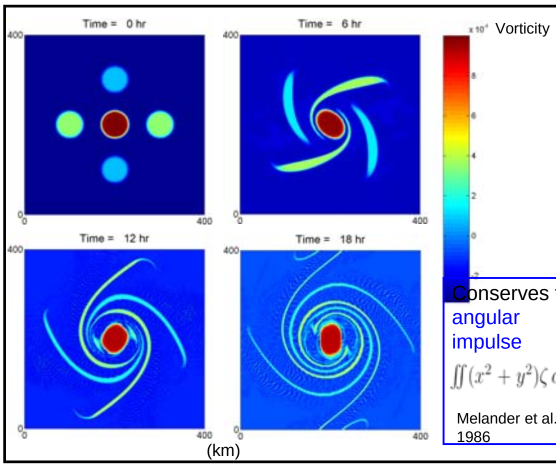


Conserves the angular impulse

$$\iint (x^2 + y^2)\zeta \, dx dy$$

Melander et al. 1986

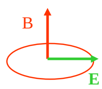
45

Electron density redistribution in experimental plasma physics

single sign charge
+
axial magnetic field
confinement

$$\mathrm{E} = -\nabla\psi$$

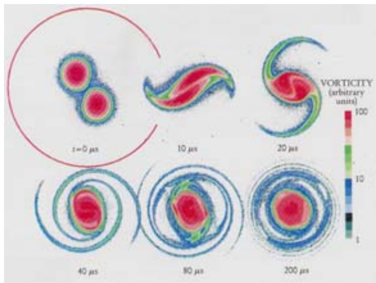
$$\nabla\cdot\mathrm{E} = -\nabla^2\psi = \frac{\rho}{\epsilon}$$



$\vec{\mathrm{E}}\times\vec{\mathrm{B}}$ drift

Coriolis force

Axisymmetrization 軸對稱化



Core is protected, thin filaments from edges

46

Huang and Robinson 1998

Forcing added

Inverse energy cascade to zonal Harmonics

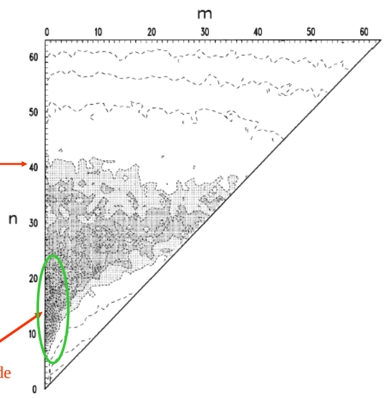


FIG. 2. Energy spectrum $E(m, n)$ of an ensemble mean at day 80 of 10 decaying turbulence experiments. The magnitude of the spectrum is normalized by the maximum value on the map. Contour levels are 0.0001, 0.001, 0.01, 0.1–0.9 with increment 0.1. Area with $E(m, n) > 0.1$ is lightly shaded, $E(m, n) > 0.2$ heavily shaded.

Rotational period 9.84hr

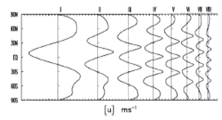


FIG. 4. Time-mean zonal-mean zonal wind profiles for cases I–VIII in Table 1 (the eight open circles in Fig. 3). Each grid on the abscissa represents 1 m s⁻¹.

Huang and Robinson 1998

Alternating Zonal Structures

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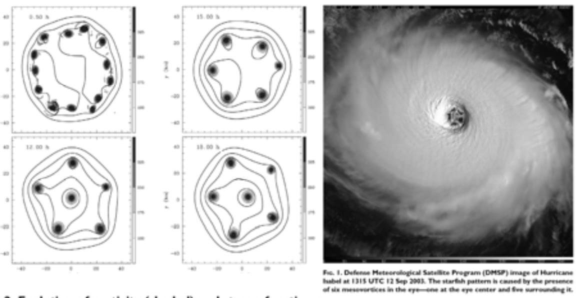

FIG. 2. Evolution of vorticity (shaded) and streamfunction contours (bold) for the numerical experiment of Kossin and Schubert (2001). Values along the label bar are in units of 10^{-4} s^{-1}. The shape of the streamlines transitions from a pentagon to a hexagon and back to a pentagon over 6 h.

MESOVORTICES IN HURRICANE ISABEL

BY JAMES P. KOSSIN AND WAYNE H. SCHUBERT

Importance of Asymmetric Vorticity Dynamics

A case where MPI theory failed! (Montgomery 2006)

49

Bowmen and Mangus (1993)

臭氧洞衛星觀測

Observations of deformation and mixing of the total ozone field in the Antarctic polar vortex

核心空氣被渦旋鎖住

細絲帶

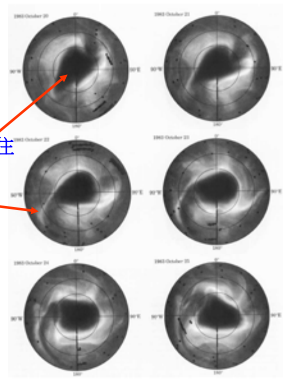

Fig.1: Daily TOMS images of total ozone in the Southern Hemisphere for six consecutive days in October 1983. Latitude circles are drawn at 40˚,60˚, and 80 ˚S. The outermost latitude is 20 ˚S.

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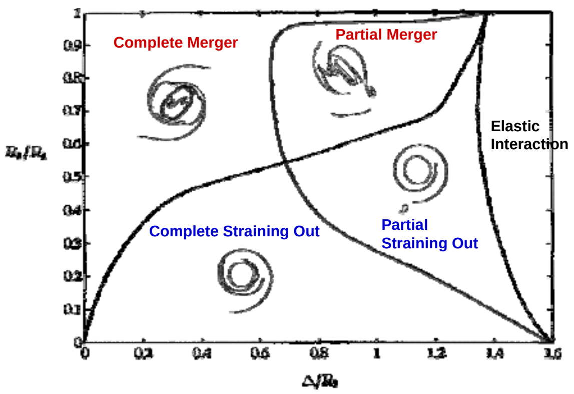

(Adapted from Dritschel and Waugh 1992.)

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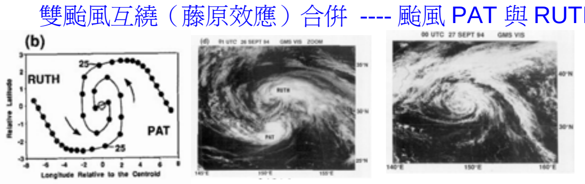
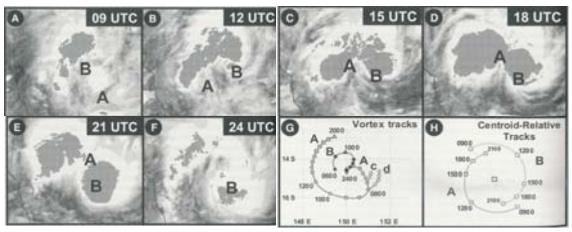
雙颱風互繞（藤原效應）合併 ---- 颱風 PAT 與 RUTH



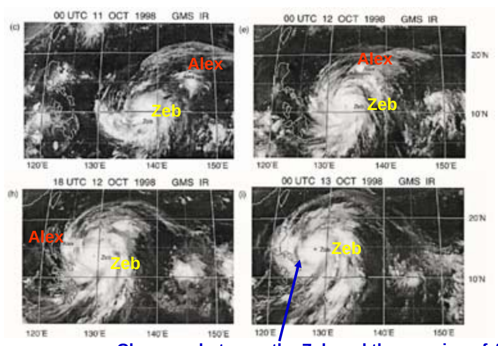
中尺度對流系統互繞

Simpson et al. 1998



颱風渦旋合併動力探討研究

Kuo, H.-C., G. T.-J. Chen, and C.-H. Lin, 2000: Merging processes of tropical cyclone Zeb and Alex. *Mon. Wea. Rev.*, **128**, 2967-2975.



Clear gap between the Zeb and the remains of Alex

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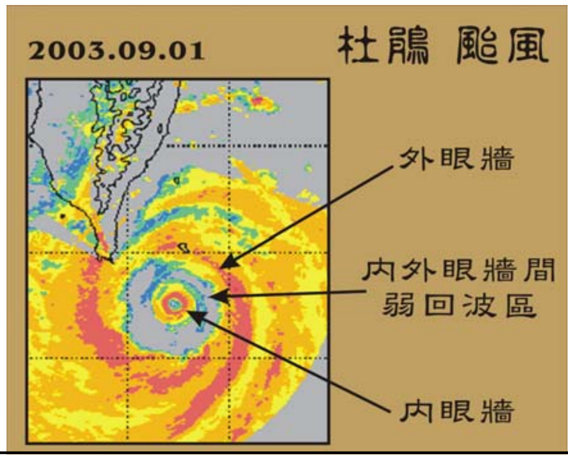
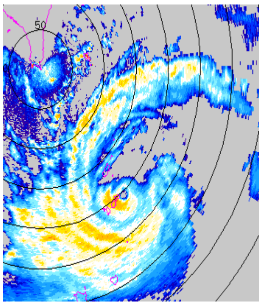
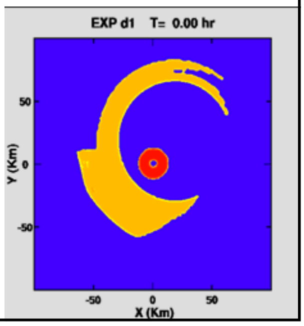


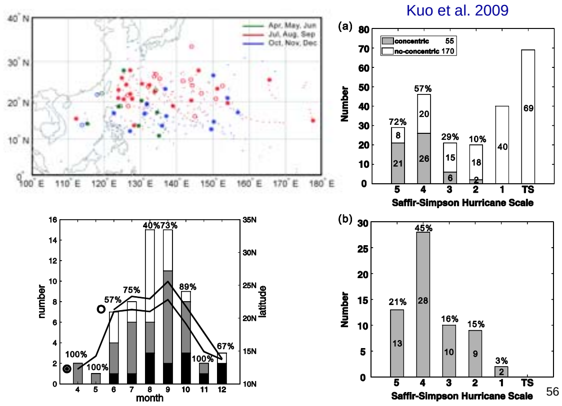
Typhoon Lekima (2001)

0935-1935 LST





Kuo et al. 2009



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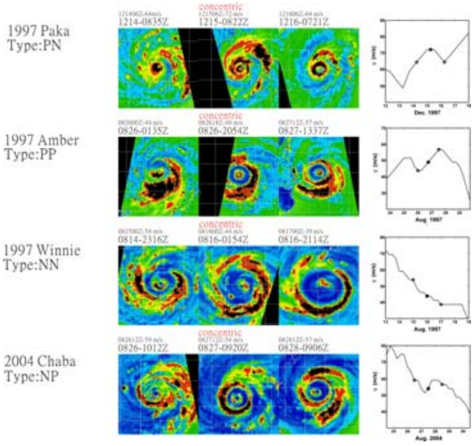


PN
+-

PP
++

NN
--

NP
-+

57

Kuo et al. 2008

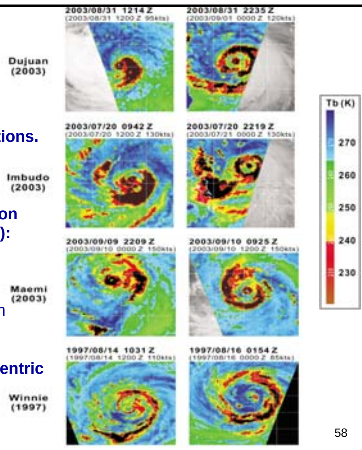
Examples of asymmetric → symmetric concentric formations.

~ 12 hours.

Initial Δ (outer deep convection region - vortex core distance):

Typhoon Dujuan: nearly 0 km
Typhoon Imbudo nearly 50 km
Typhoon Maemi: nearly 100 km
Typhoon Winnie nearly 260km

A wide range of radii of concentric eyewalls



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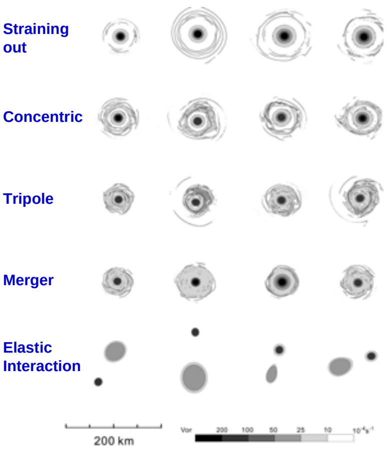
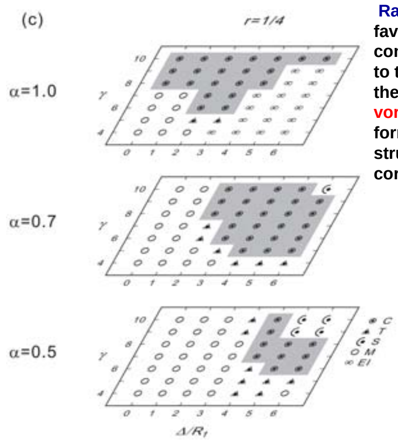
Examples of the vorticity field at hour 12, showing different classifications of binary vortex interactions involving a skirted core vortex.





Rankine vortex (α = 1.0) favors the formation of a concentric structure closer to the core vortex, while the α = 0.7 and α = 0.5 vortices favor the formation of concentric structures farther from the core vortex.

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Kuo et al. 2008 MWR

Terwey and Montgomery, June JGR 2008

D12112 TERWEY AND MONTGOMERY: MODELED SECONDARY EYEWALL FORMATION D12112

Table 1. List of Secondary Eyewall Formation Hypotheses With Summary of Relevance to our Modeled Hurricanes[a]

Authors	Hypothesis Summary	Relevance to Current Model Results	Type
Willoughby et al. [1982] borrowing from the squall line research of *Zipser* [1977]	Downdrafts from the primary eyewall force a ring of convective updrafts.	Few downdraft-forced updrafts during this time in the simulations.	O
Willoughby [1979]	Internal resonance between local inertia period and asymmetric friction due to storm motion.	No systematic storm motion in the simulated storms.	A
Hawkins [1983]	Topographic effects	No topographic forcing in the simulations.	O
Willoughby et al. [1984]	Ice microphysics	"Warm-rain" (no-ice) sensitivity case also produces secondary eyewall.	A
Molinari and Skubis [1985] and *Molinari and Vallaro* [1989]	Synoptic-scale forcings (e.g., inflow surges, upper-level momentum fluxes)	No synoptic-scale forcings in the simulations	O
Montgomery and Kallenbach [1997], *Camp and Montgomery* [2001] and *Terwey and Montgomery* [2003]	Internal dynamics-axisymmetrization via sheared vortex Rossby wave processes; collection of wave energy near stagnation or critical radii	Possible explanation	N
Nong and Emanuel [2003]	Sustained eddy momentum fluxes and WISHE feedback	Possible explanation	A
Kuo et al. [2004, 2008]	Axisymmetrization of positive vorticity perturbations around a strong and tight core of vorticity.	Possible explanation	N

[a]The type column refers to the type of model or observations that were used to formulate the hypothesis. O stands for observationally-based; A stands for axisymmetric model; N stands for nonaxisymmetric model.

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Numerical Method

- Grids Method → { Finite Difference, Finite Volume }
- Series Method → { Finite Element, Spectral Method }

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Spectral Method $f = \sum a_n \phi_n$

1. Completeness（完整性） → Sufficient condition
2. Orthogonality（正交性）
3. Speed of convergence（收斂速度）
4. Fast Transform（快速轉換）

(2–4) } Application

Sturm-Liouville equation

$$L\phi(x) = -\frac{d}{dx}(p(x)\phi'(x)) + q(x)\phi(x) = \lambda W(x)\phi(x)$$

with suitable boundary conditions and restrictions on functions $p(x)$, $q(x)$, $W(x)$, we have a countably infinite set of solutions $\{\phi_n(x)\}_{n=0}^{\infty}$ corresponding to discrete eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$

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Series Expansion Method

$$\frac{\partial u}{\partial t} + Lu = 0$$

$$u_N = \sum^{N} \hat{u}_j(t)\Phi_j(x)$$

$$R(x,\hat{u}_j) = \frac{\partial u_N}{\partial t} + Lu_N$$

$$(R(x,\hat{u}_m), \Phi_n(x)) = 0$$

Residual orthogonal to the basis function,
Smallest error in the least square sense;

$$a_{mn} = (\Phi_m(x), \Phi_n(x))$$

$$b_{mn} = (L\Phi_n(x), \Phi_m(x))$$

$$a_{mn}\frac{d\hat{u}_n}{dt} + b_{mn}\hat{u}_n = 0$$

$$a_{mn} = \delta_{mn}$$

Orthogonal; no matrix solving

Finite element; tridiagonal matrix

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Fourier, Jean Baptiste Joseph

1768-1830

Heat emission or diffusion (by IR)

His calculations showed a very cold surface (No green house effect)

$$f(x) = \sum \hat{f}_k e^{ikx}$$

$$\hat{f}_k = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}_k e^{ikx}dx$$

$$\hat{f}_k = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx$$

1807 at age 39; argued with Lagrange and Laplace on the representation of a triangle wave with cosine and sine function.

f(x) does not have to be analytical;
f(x) does not have to be periodic.

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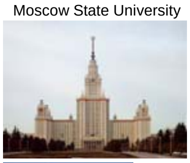
Pafnuty Lvovich Chebyshev

Wikipedia

Russian mathematician
(1821～1894)

Moscow State University



Saint Petersburg State University

Main contributions:

{ Probability, Statistics, Number theory }
→ Chebyshev's inequality
→ Bertrand-Chebyshev theorem

Chebyshev polynomials
→ Chebyshev filter

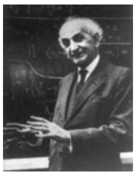
66

Cornelius Lanczos

Wikipedia

Hungarian mathematician & physicist (1893～1974)

1928 ～ 1929: He served as an assistant to Albert Einstein.



Main Contributions:
- General relativity
- Quantum mechanics
- Applied and computational mathematics
  - → Fast Fourier Transform (FFT)
  - → Chebyshev Tau method
  - → ill-posed problems

Technical University of Budapest → University of Freiburg → Purdue University → Theoretical Physics Department at the Dublin Institute (1952 ~ 1974)

67

Speed of Convergence -- Efficiency

$$f = \sum a_n \phi_n$$

$$a_n = < f, \phi_n > w = \int_a^b f(x)\phi_n(x)W(x)dx = \frac{1}{\lambda_n}\int_a^b f(x)\{-[p(x)\phi_n'(x)]' + q(z)\phi_n(x)\}dx$$

Integration by parts twice, we have

$$a_n = \frac{1}{\lambda_n}[p(f'\phi_n - f\phi_n')]_a^b + \frac{1}{\lambda_n}(\phi_n, \frac{Lf}{w})_w$$

Boundary term

If boundary term does not vanish

$$a_n \propto O(\frac{1}{\lambda_n})$$ ***algebraic convergence***

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Exponential Convergence

$$a_n = \frac{1}{\lambda_n}[p(f'\phi_n - f\phi_n')]_a^b + \frac{1}{\lambda_n}(\phi_n, \frac{Lf}{w})_w$$

0

If f is p times differentiable,
we can do integration by parts p times.

$$a_n \propto O(\frac{1}{\lambda_n^{\ p}}) \ , \ p \text{ sufficiently large}$$

Exponential Convergence

Chebyshev Equation – Sturm-Liouville Singular Problem

$$P(a) = P(b) = 0 \longrightarrow P(f'\phi_n - f\phi_n')|_a^b = 0$$

the speed of convergence depends
only on the smoothness of the function.

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Efficient Methods：
O(N) operations for O(N) degrees of freedom

- Matrix operation
  Ax = y **O(N²)**
- Inner Product **< u , ϕ_n >** **O(N²)**
  FFT , Chebyshev Transform **O(N)**
- Gaussian Elimination
  A^{-1}b = x **O(N³)**
- Relaxation (Gauss-Seidel method)
  ∇^{-2} **y = x** **O(N⁴)** for 2D **O(N²)** degrees of freedom

Accuracy: same CPU time, more accurate solution
Efficiency: same accuracy, less CPU time

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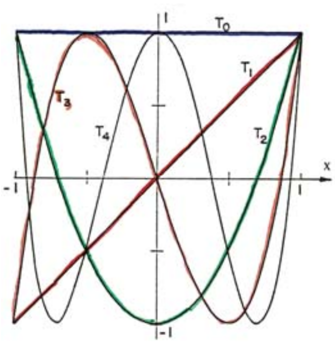
Chebyshev Polynomials

$$\begin{cases} T_n(\cos\theta) = \cos n\theta \\ x = \cos\theta \end{cases}$$

Recurrence Formula:

$$T_{n+1}(x) + T_{n-1}(x) = 2xT_n(x)$$

$$T_n(-1) = (-1)^n$$
$$T_n(1) = 1$$



Fast Chebyshev Transform

Transform pair is:

$$\begin{cases} \hat{u}_k = < u, \phi_k > & \text{physical space to spectral space} \\ u = \sum \hat{u}_k \phi_k & \text{spectral space to physical space} \end{cases}$$

Chebyshev Polynomials $\begin{cases} T_n(\cos\theta) = \cos n\theta \\ x = \cos\theta \end{cases}$

- Could take advantage of Fast Fourier Transform (FFT) (Cooley and Tukey, 1965)

General Transform $\begin{cases} 1D & O(N^2) \\ 2D & O(N^3) \end{cases}$

Fast Transform $\begin{cases} 1D & O(N \ln N) \\ 2D & O(N^2 \ln N) \end{cases}$

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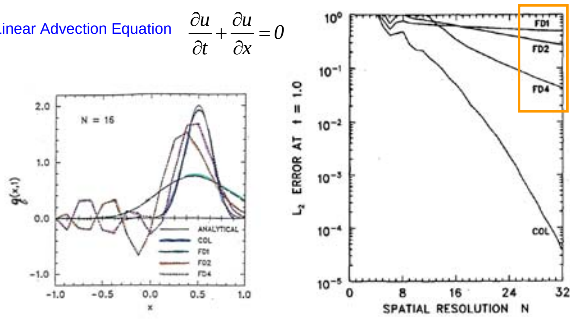
Performance of Chebyshev Collocation Method

Fulton & Schubert (1987 a)

Linear Advection Equation $\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}=0$



COL error exponential convergence

FD error algebraic convergence

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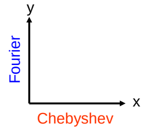
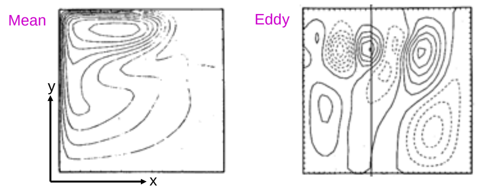
Application of Chebyshev Spectral Method in Oceanic Modeling

- Haidvogel (1976)

With pseudospectral method, employing an orthogonal expansion in Fourier and Chebyshev functions, to investigate the sensitivity and predictability of mesoscale eddies in an idealized model ocean.



Stream function



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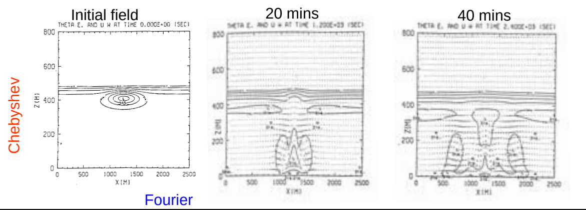
Application of Chebyshev Spectral Method in Atmospheric Modeling

- Kuo & Schubert (1988)

Applied the Fourier-Chebyshev method in a Boussinesq nonhydrostatic model to study the entrainment instability of marine boundary layer stratocumulus.



Domain Decomposition

Original Problem:

$\overline{\qquad\qquad\qquad}$ N

Domain Decomposition:

1) Same degree of freedom
   less time
   increase *Efficiency*

   $\frac{N}{4}$ $\frac{N}{4}$ $\frac{N}{4}$ $\frac{N}{4}$

2) Same time
   more degree of freedom
   increase *Accuracy*

   N N N N

PC cluster

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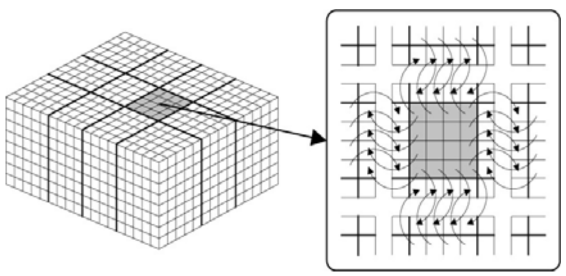

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Burger's equation:

Advection, Diffusion, Non-linear shock formation

$$\frac{\partial u}{\partial t}+(\bar{u}+u)\frac{\partial u}{\partial x}=\frac{\partial^2 u}{\partial x^2}$$

MODEL PROBLEMS

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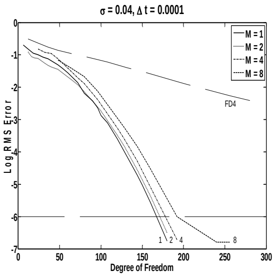
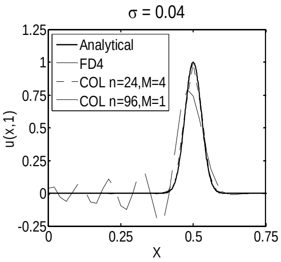
Advection Equation with Chebyshev Domain Decomposition





Exponential Convergence

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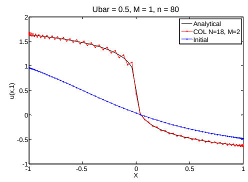
Inviscid Burger's Equation

- Atan IC
- Analytic sol. obtained by fixed point iteration (tol. = 10^{-12})
- BCs are give as analytic sol. at boundaries
- Scale collapse at t=1
- Aim to find error confinement

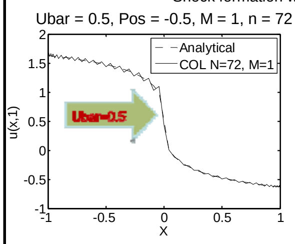
$$\begin{cases} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \varepsilon \quad (\varepsilon = 1 \text{ during all tests}) \\ u(-1,t) = \bar{u} - \tan^{-1}(x - u(-1,t)t - x_0) \\ u(1,t) = \bar{u} - \tan^{-1}(x - u(1,t)t - x_0) \\ u(x,0) = \bar{u} - \tan^{-1}(x - x_0) \end{cases}$$

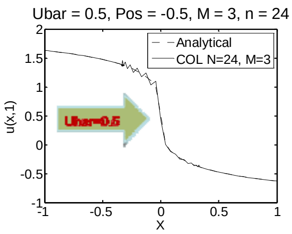
$$u(x,t) = \bar{u} - \tan^{-1}(x - u(x,t)t - x_0)$$

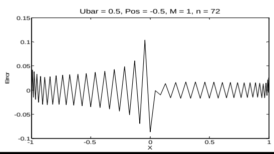


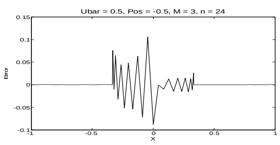
Inviscid Burger's Equation

Shock formation with advection





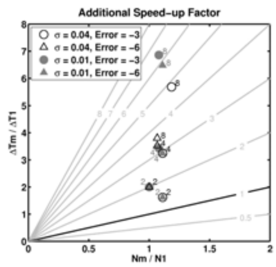




Additional Speed-up Factor

$$\text{Speedup} = M\frac{N_1\,\Delta t_M}{N_M\,\Delta t_1} = M \times \text{Additional Speedup Factor}$$

- Speed-up
- $N_M/N1$ as x
- $\Delta t_M/\Delta t_1$ as y
- Slope = Additional Sp!
- As large as 6!!
- Max. 48 times faster!!



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Shallow Water Equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv + \frac{\partial h}{\partial x} = 0 \qquad (1)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu + \frac{\partial h}{\partial y} = 0 \qquad (2)$$

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} + (\bar{h} + h)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = Q \qquad (3)$$

where $Q(x,y,t) = q_0 \exp\left[-\left(\frac{x - x_c}{x_0}\right)^2 - \left(\frac{y - y_c}{y_0}\right)^2\right] 4t^2 t_0^{-2} e^{-2t/t_0}$

$q_0 = 6250 m^2 s^{-2}$, $x_0 = y_0 = 200km$,

$t_0 = 6$ hours $= 21600$ sec, $(x_c, y_c) = (1000, -1000)$

$\bar{h} = c^2 = 2500 \frac{m^2}{s^2}$

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Shallow Water Equation

- Initial Condition

$$u(x,y,0) = -U\cos\left[\pi\left(\frac{y - y_a}{y_b - y_a}\right)\right],$$

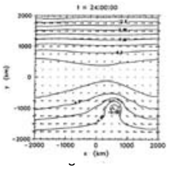
$$v(x,y,0) = 0$$

$$\frac{\partial h}{\partial y}(x,y,0) = -fu$$

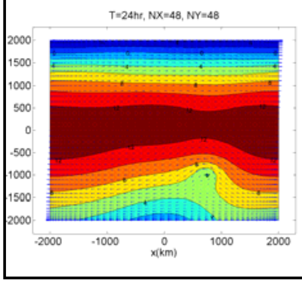
where $U = 75\ ms^{-1}$, $y_a = -2000$ km, $y_b = 2000$ km

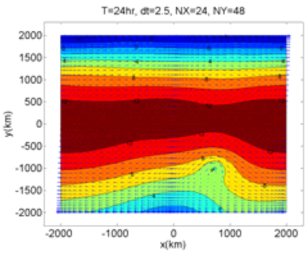
- Domain Decomposition : Overset

Domain Decomposition Numerical Results

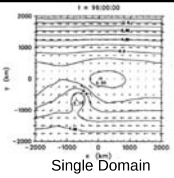


Domain Decomposition 2*1



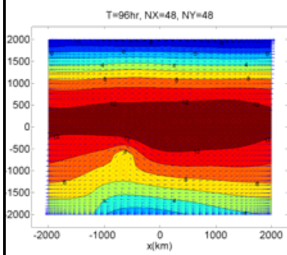


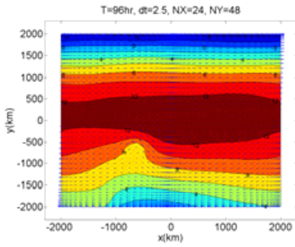
Domain Decomposition Numerical Results



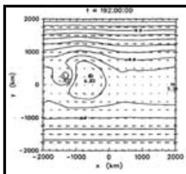
Single Domain

Domain Decomposition 2*1



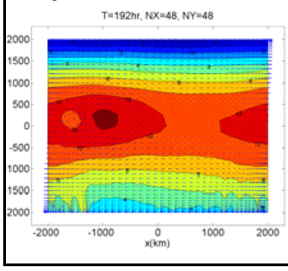


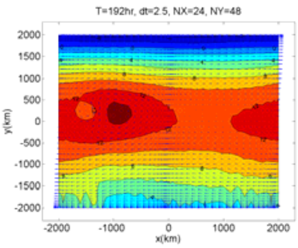
Domain Decomposition Numerical Results



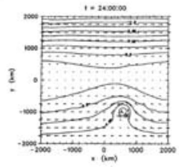
Single Domain

Domain Decomposition 2*1



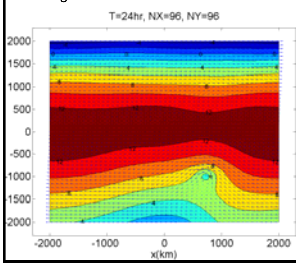


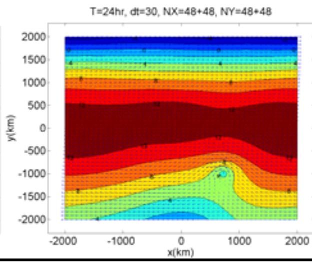
Domain Decomposition Numerical Results



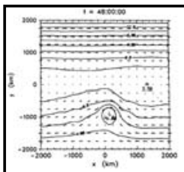
Single Domain

Domain Decomposition(2*2)



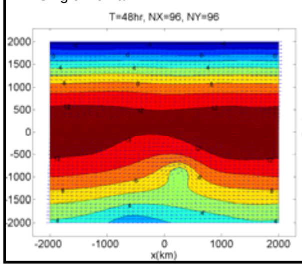


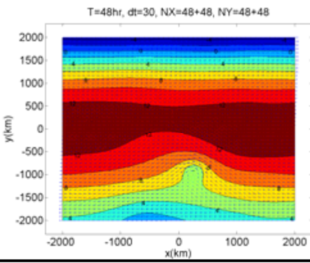
Domain Decomposition Numerical Results



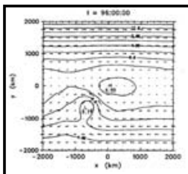
Single Domain

Domain Decomposition(2*2)



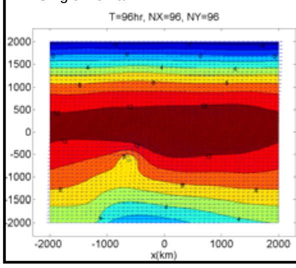


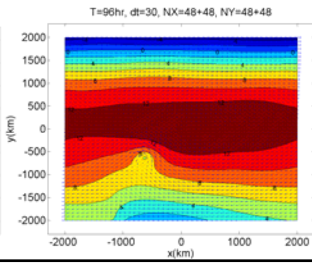
Domain Decomposition Numerical Results

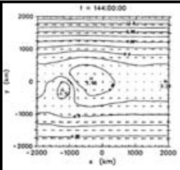


Single Domain

Domain Decomposition(2*2)



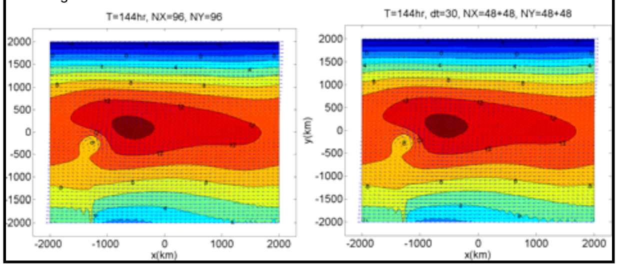


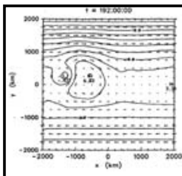


Domain Decomposition Numerical Results

Single Domain

Domain Decomposition(2*2)

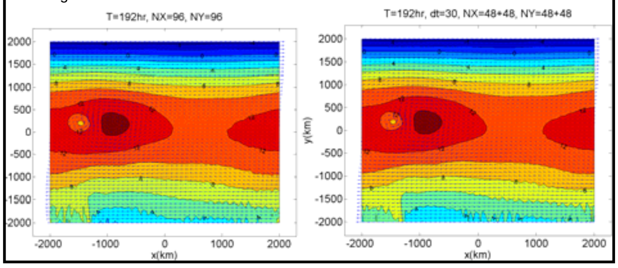


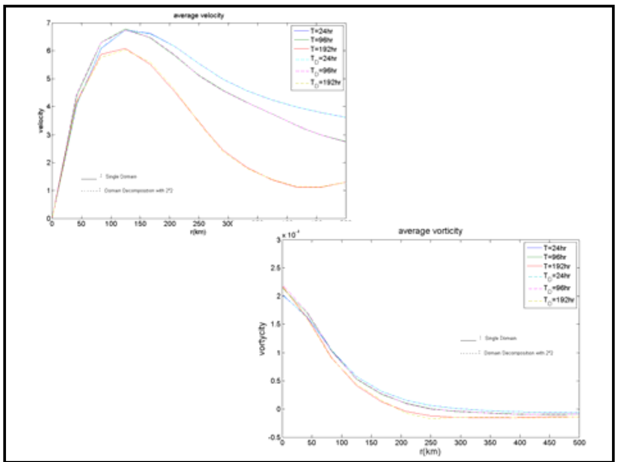


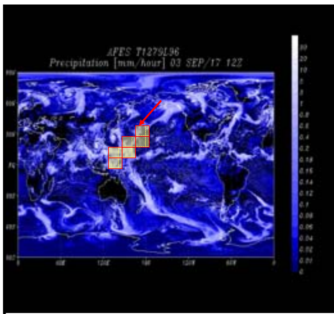
Domain Decomposition Numerical Results

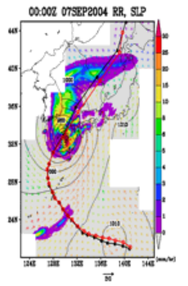
Single Domain

Domain Decomposition(2*2)









Example of Domain Decomposition: CReSS Model (Tiling) 7 days simulation

Precipitation rate (mm/hr) at 5 days from initial time.

Red line: JMA best track of Typhoon 18.

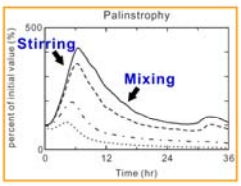
A coffee lover's dream:
The best part of waking up, is the vortex in your cup!

$$\frac{D\theta}{Dt} = \frac{\partial\theta}{\partial t} + \vec{V}\cdot\nabla\theta = \nu\nabla^2\theta$$

$$C = \frac{1}{2}\int \nabla\theta\cdot\nabla\theta\, dV$$

$$\frac{dC}{dt} = \int(\vec{V}\cdot\nabla\theta)\nabla^2\theta\, dV - \nu\int(\nabla^2\theta)\, dV$$

Stirring **Mixing**



Coffee with white

Transform Pair $u = \sum \hat{u}_k \phi_k$

$\hat{u}_k = \langle u, \phi_k \rangle$

To get $\hat{u}_k$ **in computer,**

$$\hat{u}_k = \langle u, \phi_k \rangle = \sum_j u(x_j)\phi_k(x_j)\Delta x_j$$

Let $u(x_j) = \underline{u}$ **and** $\hat{u}_k = \underline{\hat{u}}$

then A **matrix has components** $\phi_k(x_j)$

$\underline{\hat{u}} = A\underline{u}$ **matrix multip** $O(N^2)$

2D model $O(N^3)$

Fast Transform
(FFT, Fast Chebyshev Transform)

$1D \quad O(N\ln N)$

$2D \quad O(N^2 \ln N)$

$\phi_k(x)$ **from Sturm-Liouville equations**

(1) orthonormal in the inner product

$$(\phi_i,\phi_j)_w = \int_a^b \phi_i(x)\phi_j(x)w(x)dx = \delta_{ij}$$

(2) $\phi_k(x)$ form a complete set

Example: $-\frac{d}{dx}(p(x)\frac{d\phi(x)}{dx}) + q(x)\phi(x) = \lambda W(x)\phi(x)$

If $p(x) = 1 - x^2$, $q(x) = 0$, $-1 \le x \le 1$

$\Rightarrow \frac{d}{dx}((1-x^2)\frac{d\phi}{dx}) + \lambda\phi = 0$

Lgendre function

If $p(x) = 1$, $q(x) = 0$, $0 \le x \le 2\pi$

$\Rightarrow \frac{d^2\phi}{dx^2} + \lambda\phi = 0$

Fourier series

Example: $-\frac{d}{dx}(p(x)\frac{d\phi(x)}{dx}) + q(x)\phi(x) = \lambda W(x)\phi(x)$

If $p(x) = (1-x^2)^{1/2}$, $q(x) = 0$, $w(x) = (1-x^2)^{-1/2}$, $-1 \le x \le 1$

$\Rightarrow \frac{d}{dx}\left[(1-x^2)^{1/2}\frac{d\phi}{dx}\right] + \lambda(1-x^2)^{-1/2}\phi = 0$

Chebyshev series

- **Fourier, Legendre functions have been used in global spectral model**
- **Chebyshev functions are used in the limited area spectral modeling**

Finite Difference Method

FD-1 $\quad v_j = \frac{u_{j+1} - u_j}{\Delta x}$

FD-2 $\quad v_j = \frac{u_{j+1} - u_{j-1}}{2\Delta x}$

FD-4 $\quad v_j = \frac{4}{3}\frac{u_{j+1} - u_{j-1}}{2\Delta x} - \frac{1}{3}\frac{u_{j+2} - u_{j-2}}{4\Delta x}$

FD-6 $\quad v_j = \frac{3}{2}\frac{u_{j+1} - u_{j-1}}{2\Delta x} - \frac{3}{5}\frac{u_{j+2} - u_{j-2}}{4\Delta x} + \frac{1}{10}\frac{u_{j+3} - u_{j-3}}{6\Delta x}$

Define $C_n = \begin{cases} 2 & n=0 \\ 1 & n>0 \end{cases}$ **then** $\frac{2}{\pi C_n}(T_m, T_n) = \begin{cases} 1 & m=n \\ 0 & m \ne n \end{cases}$

$$\theta(x,t) = \sum_{m=0}^{\infty} \hat{\theta}_m(t) T_m(x)$$

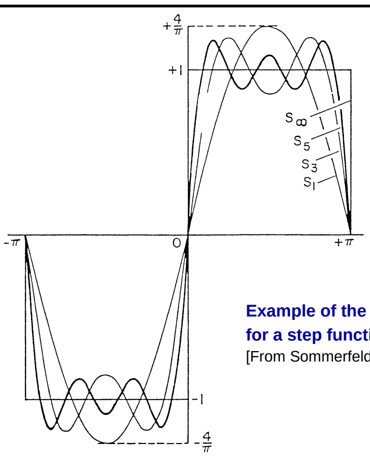
$$< \theta(x,t),\ T_n(x) > = \sum_{m=0}^{\infty} \hat{\theta}_m(t) < T_m(x),\ T_n(x) >$$

$$= \frac{\pi C_n}{2}\hat{\theta}_n(t)$$

Transform pair is

$\theta(x,t) = \sum_{n=0}^{\infty} \hat{\theta}_n(t) T_n(x)$ — *spectral space to physical space*

$\hat{\theta}_n(t) = \frac{2}{\pi C_n} < \theta(x,t), T_n(x) >$ — *physical space to spectral space*



Example of the Gibbs phenomenon for a step function.

[From Sommerfeld (1949).]

$$\theta(x) = \sin[M\pi(x+a)] = \sum_{n=0}^{\infty} \hat{\theta}_n T_n(x)$$

Define L2 error as

$$\left\{\int_{-1}^{1}\left[\theta(x) - \sum_{n=0}^{N} \hat{\theta}_n T_n(x)\right]^2 dx\right\}^{1/2}$$

Rule:

Chebyshev expansions converge rapidly when at least π polynomial are retained per wavelength

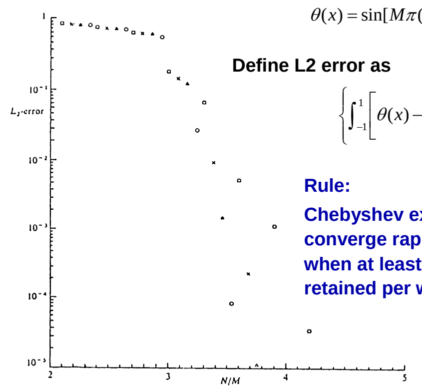
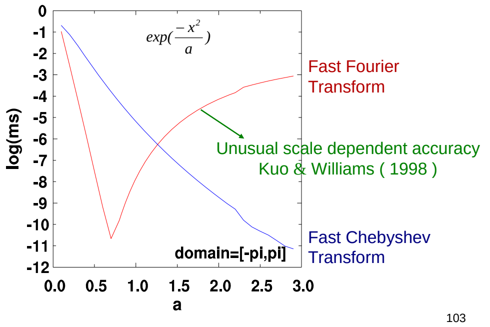


FIG. 3.7. *A plot of the L_2-error in the Chebyshev series expansion (3.41) of sin ($M\pi x$) truncated after $T_N(x)$ versus N/M. The various symbols represent:* □ $M = 10$; × $M = 20$; △ $M = 30$; ○ $M = 40$. *Observe that the L_2-error approaches zero rapidly when $N/M > \pi$.*

Chebyshev Transform *vs.* FFT



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Theoretical Speedup

$$SP(n) = \frac{s+p}{s+p/n} = \frac{s/p + 1}{s/p + 1/n}$$

n: number of working processors

s: time spent by the sequential portion of the code

p: time spent by the parallel portion of the code

	$SP(n)$	
$s/p \to 0$	n	Domain Decomposition MPI
$s/p \to 1$	$\frac{2n}{n+1}$	
$s/p \to \infty$	1	

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Reliable And Efficient Methods Exist
More Issues Need To Be Considered Other Than Efficiency !!

- **Geostrophic Adjusment**
  C-grid for Finite Differences
  Z-grid for Spectral and Finite Element
- **Axisymmetrization Dynamics**
  $\overline{r^2}$ conserved
- **Selective Decay (Statistical Dynamics)**
  - Improvement over simple ∇^2 diffusion in global or regional or hurricane models
    - **Anticipated potential vorticity method**
      - Sadourny and Basdevant 1985
      - Arakawa and Hsu 1990
      - Kazantsev et al. 1998 (Boltzmann mixing entropy maximized under energy conservation constraint)
  - Coherent Structure vs 2D Turbulences

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- **Conservations :**
  Enstrophy, Vorticity, Kinetic Energy, Available Potential Energy, Water Substance, angular momentum etc
- **Topography**
  hurricane spin-down, turbulence structure
- **Positive Definite Method**
- **Hybrid θ – σ coordinate**
  (quasi-Lagrangian vertical corrdinate)

High Resolution Direct Simulations
Cumulus Parameterization Abandoned?!
Direct simulations of Micro-states

Collective Effects, Scale Interactions
Statistical Physics, Macro Model
Efficient Numerical Methods

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Chebyshev Collocation & Tau Method

- Chebyshev Collocation Method: Doing derivation in spectral space, then inverse transform to physical space to do integration, applying boundary conditions in physical space. (*pseudospectral*)

- Chebyshev Tau Method: Doing all derivation, integration, and applying boundary conditions in spectral space, after all, inverse transform to physical space.
(Lanczos, 1938b, 1952c,d, 1956)

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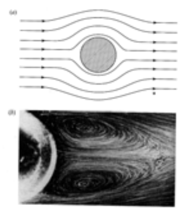
1717-1783

D'Alembert Paradox



$t_v >> t_v,\quad \frac{R}{v} >> \frac{R^2}{\nu},\quad or\ \frac{vR}{\nu} = \mathrm{Re} << 1.$

D'Alembert Solution of the Wave Equation
[f(x + ct) and f(x-ct)]

Re small viscosity important
Re large viscosity unimportant

Atmospheric Motion first expressed mathematically
(Won the 1746 Berlin Academy's Award; aqua-planet Endorsement of Euler)

Solar and Lunar Force Cause the Atmopheric Motion

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Development of Thermodynamics 熱力學
19 century

第一定律 能量作功，能量守恆
First law: Energy is what makes it go and energy is conserved.
$\triangle Q = \triangle U + WORK$
Second law: Entropy tells it where to go!
第二定律 時間之矢，自然單向

Joule, Rudolf Clausius, Lord Kelvin and others

宏觀 微觀
Macro --- Micro
Classical and Statistical Thermodynamics
統計熱力學

Ludwig Boltzmann, 1844-1906, whose work led to an understanding of the macroscopic world on the basis of molecular dynamics.

S = k Log W

雲微物理
Precipitation

Enthalpy
Entropy
Gibbs Free energy

Planck, Unwilling Revolutionary: the idea of quantization

1900

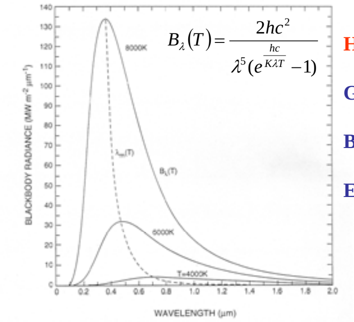


Figure 8.7 Spectra of emitted intensity $B_\lambda(T)$ for blackbodies at several temperatures, with wavelength of maximum emission $\lambda_m(T)$ indicated.

Hall of Fame in Science

Gravitational Law

Blackbody Radiation

E= MC^2

科氏力(18 19)

Momentum Conservation (18)

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}-fv=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\nabla^2 u$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+fu=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\nabla^2 v$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g+\nu\nabla^2 w$$

Mass conservation (18)

$$\frac{\partial \rho}{\partial t}+\frac{\partial u\rho}{\partial x}+\frac{\partial v\rho}{\partial y}+\frac{\partial w\rho}{\partial z}=0$$

Energy conservation (19)

$$\frac{\partial \theta}{\partial t}+u\frac{\partial \theta}{\partial x}+v\frac{\partial \theta}{\partial y}+w\frac{\partial \theta}{\partial z}=Q$$

Equation of State(17,18,19)

$$p=\rho R_a T, \quad \theta=T\left(\frac{p_0}{p}\right)^{\frac{R_a}{c_p}}$$

(19)

Radiation
大氣輻射
(19,20)
Moisture
Latent heat

雲物理
(19,20)

問蒼茫大氣，誰主浮沈？
質量、動量、能量與大氣狀態方程式

Lewis Fry Richardson, 1881–1953.

During WWI, Richardson computed by hand the pressure change at a single point.

It took him two years !

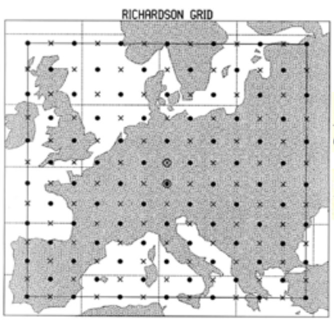
His 'forecast' was a catastrophic failure:

$\Delta p = 145$ hPa in 6 hours

His method was unimpeachable.

So, *what went wrong?*

Peter Lynch

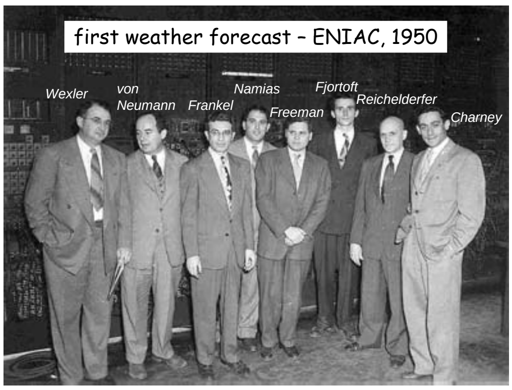


$$\frac{df}{dx} \rightarrow \frac{f(x+\Delta x)-f(x-\Delta x)}{2\Delta x}$$

$$\frac{dQ}{dt} \rightarrow \frac{Q^{n+1}-Q^{n-1}}{2\Delta t} = F^n$$

13x13=169個ODE

169 自由度



In front of the Eniac, Aberdeen Proving Ground, April 4, 1950, on the occasion of the first numerical weather computations carried out with the aid of a high-speed computer.

Nonlinear computational instability
and **the Arakawa Jacobian (1966)**

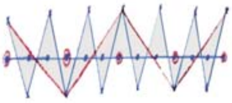
$J(\psi,\zeta)$

When the Arakawa Jacobian is used for the advection terms in the QG baroclinic model, together with the vertical differencing, the sum of kinetic energy and available potential energy is conserved, as well as potential enstrophy, in the absence of heating and friction.

➔**energy and enstrophy conservation**

➔**穩定的大氣，大氣環流之數值模式**

Nonlinear energy transfer
Aliasing error



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Orthogonality of Chebyshev series

$$(u,v)=\int_{-1}^{1}\frac{u(x)v(x)}{(1-x^2)^{1/2}}dx \qquad \textit{as inner product}$$

In particular

$$(T_m,T_n)=\int_{-1}^{1}\frac{T_m(x)T_n(x)}{(1-x^2)^{1/2}}dx$$

$$T_m(x)=\cos m\phi \quad T_n=\cos n\phi \quad x=\cos\phi$$

$$dx=-\sin\phi\, d\phi=-(1-\cos^2\phi)^{1/2}d\phi$$
$$=-(1-x^2)^{1/2}d\phi$$

$$(T_m,T_n)=\int_0^{\pi}\cos m\phi\cos n\phi\ d\phi$$

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$$\frac{D\mathbf{V}}{Dt}+2\mathbf{\Omega}\times\mathbf{V}=-\frac{1}{\rho}\nabla_z p+\nu\nabla^2\mathbf{V}.$$

$$\frac{D\mathbf{V}}{Dt}+f\mathbf{k}\times\mathbf{V}=-\nabla_p\phi+\nu\nabla^2\mathbf{V}.$$

$$\epsilon\frac{D\mathbf{V}^*}{Dt^*}+\mathbf{k}\times\mathbf{V}^*=-\nabla_p^*\phi^*+\frac{\epsilon}{\mathrm{Re}}\nabla^{*2}\mathbf{V}^*.$$

Geostrophy
Rotation Dynamics

Singular Perturbation Problems
Quasi-balanced Dynamics

Boundary Layer Dynamics
Nearly Invicid

$\epsilon=\dfrac{1/f}{L/U}$ **Rotation time scale / Advective time scale**

$Re=\dfrac{L^2/\nu}{L/U}$ **Diffusion time scale / Advective time scale**

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$$f=\sum a_k\phi_k$$
$$a_k=<f,\phi_k>w$$
$$=\int_a^b f(x)\phi_n(x)W(x)dx$$
$$=\frac{1}{\lambda_n}\int_a^b f(x)\{-[p(x)\phi_n'(x)]'+q(z)\phi_n(x)\}dx$$

Speed of convergence
➔ Efficiency
➔ Boundary condition

Integrate by parts twice, we have

$$a_n=\frac{1}{\lambda_n}[p(f'\phi_n-f\phi_n')]_a^b+\frac{1}{\lambda_n}(\phi_n,\frac{Lf}{w})_w$$ ➔ **Boundary term**

If boundary term not vanish

$a_n=O(\frac{1}{\lambda_n})$ ***algebraic convergence***

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Nonsingular Problem $P>0 \quad \text{on}\,[a,b]$

for $P(f'\phi_n-f\phi_n')\big|_a^b=0$

we need $f'\phi_n-f\phi_n'\big|_a^b=0$

➔ **Periodic domain**

➔ **Exponential Convergence**

※ **This is the case for Fourier Series**

Singular Problem $P(a)=P(b)=0$

then $P(f'\phi_n-f\phi_n')\big|_a^b=0$

Regardless of the behavior of f(x) near the boundary a, b

➔**Exponential Convergence**

※ **This is the case for Legendre and Chebyshev polynomials**

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If $\frac{1}{\lambda_n}[p(f'\phi_n-f\phi_n')]_a^b=0$

and ϕ_n **is p times differentiable**

We can do integration by parts p times

$$a_n<O(\frac{1}{\lambda_n^p})$$

➔ ***Exponential convergence***

When boundary terms vanished,
the speed of convergence
depends on the smoothness of the function.

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