Problems for the dot product, the cross product, and the geometry

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High School Science Project
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(1) Use the scalar projection to show that the distance from a point $P_1(x_0, y_0)$ to the line of $ax + by + c = 0$ is

$$\frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}.$$

Use this formula to find the distance from the point $(-2, 3)$ to the line $3x - 4y + 5 = 0$.

(2) Use the scalar projection to show that the distance from a point $P_1(x_0, y_0, z_0)$ to the plane of $ax + by + cz + d = 0$ is

$$\frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Use this formula to find the distance from the point $(-2, 3, 1)$ to the line $3x - 4y + 5z + 1 = 0$.

(3) If $\mathbf{r} = \langle x, y, z\rangle$, $\mathbf{a} = \langle a_1, a_2, a_3\rangle$, and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$, show that the vector equation $(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{r} - \mathbf{b}) = 0$ represents a sphere, and find its center and radius.

(4) Prove the Cauchy-Schwarz Inequality:

$$|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|.$$

(5) The Triangle Inequality for vectors is

$$|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|,$$

(a) Give a geometrical interpretation of the Triangle Inequality.

(b) Use the Cauchy-Schwarz Inequality to prove the Triangle Inequality.[hint: use $|\mathbf{a}+\mathbf{b}|^2 = (\mathbf{a}+\mathbf{b}) \cdot (\mathbf{a}+\mathbf{b})$.]

(6) The Parallelogram Law states that

$$|\mathbf{a}+\mathbf{b}|^2 + |\mathbf{a}-\mathbf{b}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2.$$

(a) Give a geometrical interpretation of the Parallelogram Law.

(b) Prove the Parallelogram Law.

(7-9) (a) Find a vector orthogonal to the plane through the points P, Q, and R, and (b) find the area of triangle PQR.

(7) $P = (1,0,0)$, $Q = (0,2,0)$, and $R = (0,0,3)$.

(8) $P = (1,0,-1)$, $Q = (2,4,5)$, and $R = (3,1,7)$.

(9) $P = (0,0,0)$, $Q = (1,-1,1)$, and $R = (4,3,7)$.

(10-11) Find the volume of the parallelepiped determined by the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

(10) $\mathbf{a} = \langle 1,0,6 \rangle$, $\mathbf{b} = \langle 2,3,-8 \rangle$, and $\mathbf{c} = \langle 8,-5,6 \rangle$.

(11) $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j}$, and $\mathbf{c} = 2\mathbf{i} + 3\mathbf{k}$.

(12) (a) Let P be a point not on the line L that passes through the points Q and R. Show that the distance d from the point P to the line L is

$$d = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|}$$

where $\mathbf{a} = \vec{QR}$ and $\mathbf{b} = \vec{QP}$.

(b) Use the formula in part (a) to find the distance from the point $P(1,1,1)$ to the line through $Q(0,6,8)$ and $R(-1,4,7)$.

(c) Compare this problem with problem (1).

(13) (a) Let P be a point not on the plane that passes through the points Q, R, and S. Show that the distance d from the point P to the plane is

$$d = \frac{|\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})|}{|\mathbf{a} \times \mathbf{b}|}$$

where $\mathbf{a} = \vec{QR}$, $\mathbf{b} = \vec{QS}$ and $\mathbf{c} = \vec{QP}$.

(b) Use the formula in part (a) to find the distance from the point $P(2, 1, 4)$ to the plane through $Q(1, 0, 0)$, $R(0, 2, 0)$, and $S(0, 0, 3)$.

(14) Prove that $(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = 2(\mathbf{a} \times \mathbf{b})$.

(15) (a) What is the equation for the line passes through the point $P(x_0, y_0)$ and perpendicular to the vector $\langle a, b\rangle$.

(b) What is the equation for the plane passes through the point $P(x_0, y_0, z_0)$ and perpendicular to the vector $\langle a, b, c\rangle$.

(16) (a) Find the angle between the lines $x + y = 3$ and $x - y = 3$.

(b) Find the angle between the planes $x + y + z = 1$ and $2x - y - z = 5$.

(c) Find the angle between the planes $x + y + z = 1$ and $2x + y + z = 5$.

(17) Show that the distance between the two parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is

$$D = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}.$$

(18) Find the equation of the planes that are parallel to the plane $x + 2y - 2z = 1$ and two units away from it.

(19) Skew lines are lines do not intersect and are not parallel. What is the geometrical interpretation of **skew lines**?

(20-21) Write to learn the following statement. [hint: see 12.5 in handout].

(20) (a) Explain that a line L in three dimensional space can be written in the **vector equation**

$$\mathbf{r} = \mathbf{r_0} + t\mathbf{v}$$

with $\mathbf{v}$ is parallel to L and each value of **parameter** t gives the position vector $\mathbf{r}$ of a point on L.

(b) The above equation can also be written as $x = x_0 + at$, $y = y_0 + bt$, and $z = z_0 + ct$, the **parametric equation**. How are the vectors $\langle a, b, c\rangle$ and $\langle x_0, y_0, z_0\rangle$ in **parametric equation** related to the **vector equation** ?

(c) Explain why the line equation can also be written as the **symmetric equation**

$$\frac{x - x_0}{a} = \frac{x - x_0}{b} = \frac{x - x_0}{c}.$$

(21) Explain why the plane equation can be written as

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = 0$$

where the $\mathbf{n}$ is orthogonal to $(\mathbf{r} - \mathbf{r_0})$ (normal to the plane). The plane equation can also be written as

$$ax + by + cz + d = 0.$$

Explain how this form is related to the vector form in (a).

(22) (a) Find the point at which the given lines intersect:

$$\mathbf{r} = \langle 1, 1, 0\rangle + t\langle 1, -1, 2\rangle$$

$$\mathbf{r} = \langle 2, 0, 2\rangle + s\langle -1, 1, 0\rangle.$$

(b) Find an equation of the plane that contains these lines.

(23) Show that the lines

$$L_1 : x = 1 + t \quad y = -2 + 3t \quad z = 4 - t$$

$$L_2 : x = 2s \quad y = 3 + s \quad z = -3 + 4s$$

are skew. Find the distance between them.[hint: see page 818].

(24) A tetrahedron is a solid with four vertices, P, Q, R, and S, and four triangle faces.

(a) Draw a schematic diagram for the tetrahedron.

(b) Let $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$, and $\mathbf{v_4}$ be vectors with lengths equal to the areas of the faces opposite the vertices P, Q, R, and S, respectively, and directions perpendicular to the respective faces and pointing outward. Show that $\mathbf{v_1} + \mathbf{v_2} + \mathbf{v_3} + \mathbf{v_4} = 0$.

(c) The volume V of a tetrahedron is one-third the distance from a vertex to the opposite face, times the area of that face. Find a formula for the volume of a tetrahedron in terms of the coordinates of its vertices P, Q, R, and S. Check your solution with the tetrahedron whose vertices are $P(1,1,1)$, $Q(1,2,3)$, $R(1,1,2)$, and $S(3,-1,2)$.

(d) Suppose the tetrahedron has a trirectangular vertex S. (This means that the three angles at S are all right angles.) Let A, B, and C be the areas of the three faces that meet at S, and let D be the area of the opposite face PQR. Using the result of (b), or otherwise, show that $D^2 = A^2 + B^2 + C^2$. (This is a three-dimensional version of the Pythagorean Theorem.)